

ON SCHUR p -GROUPS OF ODD ORDER

GRIGORY RYABOV

ABSTRACT. A finite group G is called a Schur group if any S -ring over G is associated in a natural way with a subgroup of $\text{Sym}(G)$ that contains all right translations. We prove that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, where $n \geq 1$, are Schur. Modulo previously obtained results, it follows that every noncyclic Schur p -group, where p is an odd prime, is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, $n \geq 1$.

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1. INTRODUCTION

Let G be a finite group, e the identity element of G . Let $\mathbb{Z}G$ be the integer group ring. Given $X \subseteq G$, denote the element $\sum_{x \in X} x$ of $\mathbb{Z}G$ by \underline{X} and the set $\{x^{-1} : x \in X\}$ by X^{-1} .

Definition 1.1. A subring \mathcal{A} of $\mathbb{Z}G$ is called an S -ring over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that:

- (1) $\{e\} \in \mathcal{S}$,
- (2) $X \in \mathcal{S} \Rightarrow X^{-1} \in \mathcal{S}$,
- (3) $\mathcal{A} = \text{Span}_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}\}$.

The elements of this partition are called *the basic sets* of the S -ring \mathcal{A} .

The definition of an S -ring goes back to Schur and Wielandt, they used “the S -ring method” to study a permutation group having a regular subgroup [11, 12].

Let Γ be a subgroup of $\text{Sym}(G)$ that contains the subgroup of right translations $G_{\text{right}} = \{x \mapsto xg, x \in G : g \in G\}$. Let Γ_e stand for the stabilizer of e in Γ and $\text{Orb}(\Gamma_e, G)$ stand for the set of all orbits of Γ_e on G . As Schur proved in [11], the \mathbb{Z} -submodule

$$\mathcal{A} = \mathcal{A}(\Gamma, G) = \text{Span}_{\mathbb{Z}}\{\underline{X} : X \in \text{Orb}(\Gamma_e, G)\},$$

is an S -ring over G .

Definition 1.2 (Pöschel, 1974). An S -ring \mathcal{A} over G is called *schurian* if $\mathcal{A} = \mathcal{A}(\Gamma, G)$ for some Γ with $G_{\text{right}} \leq \Gamma \leq \text{Sym}(G)$.

Definition 1.3 (Pöschel, 1974). A finite group G is called a *Schur group* if every S -ring over G is schurian.

Wielandt wrote in [13]: “Schur had conjectured for a long time that every S -ring is determined by a suitable permutation group”, or in our terms that every S -ring is schurian. However, it turns out to be false. The first counterexample found by Wielandt [12] is an S -ring over $\mathbb{Z}_p \times \mathbb{Z}_p$, where $p > 3$ is a prime. The problem of determining all Schur groups was suggested by Pöschel in [9] about 40 years ago. He proved that the cyclic p -groups are Schur whenever p is odd. Moreover, if $p > 3$ then a p -group is Schur if and only if it is cyclic. Applying this result Pöschel and Klin solved the isomorphism problem for circulant graphs with p^n vertices, where p is an odd prime [6].

Only 30 years later all cyclic Schur groups were classified in [3]. Strong necessary conditions of schurity for abelian groups were recently proved in [4]. From these conditions it follows that any abelian Schur group belongs to one of several explicitly given families. The case of non-abelian Schur

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groups was analyzed in [8]: it was proved that every Schur group G is solvable of derived length at most 2 and the number of distinct prime divisors of the order of G does not exceed 7.

In this article we are interested in abelian Schur 3-groups. From [4, Theorem 1.2, Theorem 1.3] it follows that all abelian Schur 3-groups are known except for the groups $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, where $n \geq 4$. We prove the following statement.

Theorem 1.1. *The groups $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, where $n \geq 1$, are Schur.*

Modulo previously obtained results [4, Theorem 1.2, Theorem 1.3], [8, Theorem 4.2], [10, Theorem 1], Theorem 1.1 implies the classification of Schur p -groups, where p is an odd prime:

Theorem 1.2. *A p -group G , where p is an odd prime, is Schur if and only if G is cyclic or $p = 3$ and G is isomorphic to one of the groups below:*

- (1) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$;
- (2) $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$, $n \geq 1$.

The case of Schur 2-groups was considered in [7], where the complete classification of abelian Schur 2-groups was given. From [7, Theorem 1.2] it follows that any non-abelian Schur 2-group of order at least 32 is dihedral.

In the proof of Theorem 1.1 we follow, in general, the scheme of the proof [7, Theorem 10.1]. At first, we establish that regular and rational S -rings over $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ with trivial radical are schurian. Further, we prove that an S -ring over $\mathbb{Z}_3 \times \mathbb{Z}_{3^n}$ with nontrivial radical is a “good” generalized wreath product of two smaller S -rings. In this case the S -ring is schurian by the criterion of schurity for a generalized wreath product over abelian groups [7, Theorem 10.2].

2. PRELIMINARIES

In this section we recall some definitions and facts concerned with S -rings and Schur groups. The most part of them is taken from [7]. There are references before other statements.

Definition 2.1. Let G be a finite group, \mathcal{R} a family of binary relations on G . The pair $\mathcal{C} = (G, \mathcal{R})$ is called a *Cayley scheme* over G if the following properties are satisfied:

- (1) \mathcal{R} forms a partition of the set $G \times G$;
- (2) $\text{Diag}(G \times G) \in \mathcal{R}$;
- (3) $\mathcal{R} = \mathcal{R}^*$, i. e., if $R \in \mathcal{R}$ then $R^* = \{(h, g) \mid (g, h) \in R\} \in \mathcal{R}$;
- (4) if $R, S, T \in \mathcal{R}$, then the number $|\{h \in G : (f, h) \in R, (h, g) \in S\}|$ does not depend on the choice of $(f, g) \in T$.

Let \mathcal{A} be an S -ring over G . We associate each basic set $X \in \mathcal{S}(\mathcal{A})$ with the binary relation $\{(a, xa) \mid a \in G, x \in X\} \subseteq G \times G$ and denote it by $R(X)$. The set of all such binary relations forms a partition $\mathcal{R}(\mathcal{S})$ of $G \times G$.

Lemma 2.1. $\mathcal{C}(\mathcal{A}) = (G, \mathcal{R}(\mathcal{S}))$ is a Cayley scheme over G .

The relation $R(X)$ is called *the basic relation* of the scheme $\mathcal{C}(\mathcal{A})$ corresponding to X .

Definition 2.2. Cayley schemes $\mathcal{C} = (G, \mathcal{R})$ and $\mathcal{C}' = (G', \mathcal{R}')$ are called *isomorphic* if there exists a bijection $f : G \rightarrow G'$ such that $\mathcal{R}' = \mathcal{R}^f$, where $\mathcal{R}^f = \{R^f : R \in \mathcal{R}\}$ and $R^f = \{(a^f, b^f) : (a, b) \in R\}$.

We say that S -rings \mathcal{A} over G and \mathcal{A}' over G' are *isomorphic* if the Cayley schemes associated to \mathcal{A} and \mathcal{A}' are isomorphic. Any isomorphism between these schemes is called the isomorphism from \mathcal{A} onto \mathcal{A}' . The group of all isomorphisms from \mathcal{A} to itself has a normal subgroup $\text{Aut}(\mathcal{A})$ that consists of all isomorphisms f such that $X^f = X$ for all $X \in \mathcal{S}(\mathcal{A})$; any such f is called an *automorphism* of the S -ring \mathcal{A} . An S -ring \mathcal{A} over G is schurian if and only if $\mathcal{S}(\mathcal{A}) = \text{Orb}(\text{Aut}(\mathcal{A})_e, G)$. Under a *Cayley isomorphism* from \mathcal{A} to \mathcal{A}' we mean a group isomorphism $f : G \rightarrow G'$ such that $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$. In this case \mathcal{A} and \mathcal{A}' are called *Cayley isomorphic*. Every Cayley isomorphism is an isomorphism.

Definition 2.3. Let \mathcal{A} be an S -ring over G . A subset $X \subseteq G$ is called an \mathcal{A} -set if $\underline{X} \in \mathcal{A}$.

Definition 2.4. Let \mathcal{A} be an S -ring over G . A subgroup $H \leq G$ is called an \mathcal{A} -subgroup if $\underline{H} \in \mathcal{A}$.

Lemma 2.1. Let \mathcal{A} be an S -ring over G and H be an \mathcal{A} -subgroup. Then the module

$$\mathcal{A}_H = \text{Span}_{\mathbb{Z}} \{ \underline{X} : X \in \mathcal{S}(\mathcal{A}), X \subseteq H \},$$

is an S -ring over H . In addition, if \mathcal{A} is schurian, then \mathcal{A}_H is schurian too.

Definition 2.5. Let L, U be subgroups of a group G and L be normal in U . A section U/L of G is called an \mathcal{A} -section if U and L are \mathcal{A} -subgroups.

Lemma 2.2. Let $F = U/L$ be an \mathcal{A} -section. Then the module

$$\mathcal{A}_F = \text{Span}_{\mathbb{Z}} \{ \underline{X}^\pi : X \in \mathcal{S}(\mathcal{A}), X \subseteq U \},$$

where $\pi : U \rightarrow U/L$ is the quotient epimorphism, is an S -ring over F .

In addition, if \mathcal{A} is schurian, then \mathcal{A}_F is schurian too.

The following two lemmas give well-known properties of basic sets of an S -ring.

Lemma 2.3. For any basic sets X, Y, Z of an S -ring \mathcal{A} over G denote by $c_{X,Y}^Z$ the number of distinct representations of an element $z \in Z$ in the form $z = xy$ with $x \in X$ and $y \in Y$. Then

$$|Z|c_{X,Y}^{Z^{-1}} = |X|c_{Y,Z}^{X^{-1}} = |Y|c_{Z,X}^{Y^{-1}}.$$

Lemma 2.4. For any basic sets X, Y of an S -ring \mathcal{A} over G the set XY is an \mathcal{A} -set. Moreover, if $|X| = 1$ or $|Y| = 1$ then XY is a basic set.

Lemma 2.5. Let H be an \mathcal{A} -subgroup of G and $X \in \mathcal{S}(\mathcal{A})$. Then the number $|X \cap Hg|$ does not depend on $g \in G$ with $X \cap Hg \neq \emptyset$.

Definition 2.6. Let G be a finite group and $X \subseteq G$. Then the subgroup of G defined by $\text{rad}(X) = \{g \in G | Xg = gX = X\}$ is called the radical of X .

Lemma 2.6. Let \mathcal{A} be an S -ring over G . If a set X is an \mathcal{A} -set, then the groups $\langle X \rangle$ and $\text{rad}(X)$ are \mathcal{A} -subgroups of G .

Theorem 2.1. Let X be a basic set of an S -ring \mathcal{A} over G . Suppose that $H \leq \text{rad}(X \setminus H)$ for some group H such that $X \cap H \neq \emptyset$ and $X \setminus H \neq \emptyset$. Then $X = \langle X \rangle \setminus \text{rad}(X)$ with $\text{rad}(X) \leq H \cap \langle X \rangle$. In this case H is called a separating group.

Definition 2.7. An S -ring \mathcal{A} over G is called *primitive* if G has no nontrivial proper \mathcal{A} -subgroups. Otherwise \mathcal{A} is called *imprimitive*.

Lemma 2.7. For every prime p a primitive S -ring over $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$, $a > b \geq 0$, $a > 1$, has rank 2.

Proof. Follows from the proofs of [12, Theorem 25.3, Theorem 25.5] \square

Definition 2.8. Let $K \leq \text{Aut}(G)$. Then the orbit partition $\text{Orb}(K, G)$ defines an S -ring \mathcal{A} over G . In this case \mathcal{A} is called *cyclotomic* and denoted by $\text{Cyc}(K, G)$.

For any set $X \subseteq G$ and $m \in \mathbb{Z}$ denote by $X^{(m)}$ the set $\{x^m : x \in X\}$. We say that two sets X and Y are *rationally conjugate* if there exists $m \in \mathbb{Z}$ coprime to n such that $Y = X^{(m)}$.

Further we formulate two theorems on multipliers of S -rings over abelian groups. Both of them were proved by Schur in [11].

Theorem 2.2. Let \mathcal{A} be an S -ring over an abelian group G of order n . Then $X^{(m)} \in \mathcal{S}(\mathcal{A})$ for all $X \in \mathcal{S}(\mathcal{A})$ and all $m \in \mathbb{Z}$ coprime to n .

Theorem 2.3. Let \mathcal{A} be an S -ring over an abelian group G . Then given a prime p dividing the order of G , the set

$$X^{[p]} = \{x^p : x \in X, |X \cap Hx| \not\equiv 0 \pmod{p}\},$$

where $H = \{g \in G : g^p = e\}$, is an \mathcal{A} -set for all \mathcal{A} -sets X .

The following lemma is taken from [9].

Lemma 2.8. *Let \mathcal{A} be an S -ring over cyclic p -group G , where p is an odd prime. Then for every $X \in \mathcal{S}(\mathcal{A})$ one of the following statements holds:*

- (1) $X \in \text{Orb}(K, G)$ for some $K \leq \text{Aut}(G)$;
- (2) $X = \langle X \rangle \setminus \text{rad}(X)$.

Lemma 2.9. *Let \mathcal{A} be a cyclotomic S -ring over cyclic 3-group G and S be an \mathcal{A} -section. Suppose that $\text{rad}(\mathcal{A}) = e$. Then $|\text{rad}(\mathcal{A}_S)| = 1$.*

Proof. Follows from [3, Theorem 7.3]. \square

Let $D = S \times C$, $S = \langle s \rangle$, $|s| = 3$, $C = \langle c \rangle$, $|c| = 3^n$, $n \geq 2$. Every subset X of D can be uniquely presented as the union $X = X_{0e} \cup sX_{1s} \cup s^2X_{2s^2}$, where $X_{0e}, X_{1s}, X_{2s^2} \subseteq C$. If $T \subseteq D$, we denote the set $\{t \in T : |t| \leq 3^m\}$ by T_m . Denote one of two elements of order 3 from C by c_1 , the cyclic group $\langle c_1 \rangle$ by C_1 , the elementary abelian group $S \times C_1$ by E , the basic set of the S -ring \mathcal{A} containing an element x by T_x .

Lemma 2.10. *If a basic set T of an S -ring \mathcal{A} over D contains elements a, b such that $|a| > |b| \geq 3$ then $a\{1, c_1, c_1^2\} \subseteq T$*

Proof. Set $m = 1 + \frac{|a|}{3}$, $l = 1 + \frac{2|a|}{3}$. Then $b^m = b^l = b$. By Theorem 2.2, the sets $T^{(m)}, T^{(l)}$ are basic and $T^{(m)} = T^{(l)} = T$. So $\{a^m, a^l\} = \{aa^{\frac{|a|}{3}}, aa^{\frac{2|a|}{3}}\} = \{ac_1, ac_1^2\} \subseteq T$. \square

Further we formulate and prove two lemmas that describe regular sets with trivial radical over $C = \mathbb{Z}_{3^n}$ and $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$.

Lemma 2.11. *Let $X \in \text{Orb}(M, C)$ where $M \leq \text{Aut}(C)$. Then $\text{rad}(X) = e$ if and only if $M = e$ or $M = \{e, \delta\}$, where $\delta : x \rightarrow x^{-1}$. This means that there exists $x \in C$ such that $X = \{x\}$ or $X = \{x, x^{-1}\}$.*

Proof. Follows from [2, Lemma 5.1]. \square

Lemma 2.12. *Let T be a basic set of an S -ring \mathcal{A} over D and $3^m = \min\{|t| : t \in T\}$. Suppose that $\text{rad}(T) = e$. Then there exist subsets X, Y, Y_1, Z, Z_1 of C such that $T_m = X \cup sY \cup s^2Y_1 \cup sZ \cup s^2Z_1$, $Y \cap Y_1 = Z \cap Z_1 = \emptyset$, and one of the following statements holds:*

- (1) every nonempty set $U \in \{X, Y \cup Y_1, Z \cup Z_1\}$ is singleton;
- (2) every nonempty set $U \in \{X, Y \cup Y_1, Z \cup Z_1\}$ is of the form $\{x, x^{-1}\}$, $x \in C$.

Proof. Let us present T_m as a union $T_m = T_{0e,m} \cup sT_{1s,m} \cup s^2T_{2s^2,m}$, where $T_{0e,m}, T_{1s,m}, T_{2s^2,m} \subseteq C$. Denote by K the group $\{\sigma_m : m \in \mathbb{Z}_{3^n}^*\}$, $\sigma_m : x \rightarrow x^m$, and by M the setwise stabilizer $K_{\{T\}}$ of T in K . If $a \in T_{0e,m}$ and $\alpha \in M$ then $a\alpha \in T_{0e,m}$. Thus $T_{0e,m}$ is a union of some orbits of M . Since all elements in $T_{0e,m}$ have the same order, $T_{0e,m}$ is an orbit of M . If $a \in sT_{1s,m} \cup s^2T_{2s^2,m}$ and $\alpha \in M$ then $a\alpha \in sT_{1s,m} \cup s^2T_{2s^2,m}$. Thus $sT_{1s,m} \cup s^2T_{2s^2,m}$ is a union of some orbits of M . Moreover, it is easy to check that $sT_{1s,m} \cup s^2T_{2s^2,m}$ is a union of at most two orbits of M . Therefore T_m can be presented as a union $T_m = X \cup sY \cup s^2Y_1 \cup sZ \cup s^2Z_1$, where each set $X, Y \cup Y_1, Z \cup Z_1$ is empty or equal to an orbit of M . At least one of the sets $X, Y \cup Y_1, Z \cup Z_1$ has trivial radical because otherwise $c_1 \in \text{rad}(T)$ by Lemma 2.10. Without loss of generality we assume that $X \neq \emptyset$ and $\text{rad}(X) = e$. Then from Lemma 2.11 it follows that $M = e$ or $M = \{e, \delta\}$ where $\delta : x \rightarrow x^{-1}$. Now the claim follows. \square

Now we describe two constructions producing S -rings over the group $G = G_1 \times G_2$ from S -rings \mathcal{A}_1 and \mathcal{A}_2 over G_1 and G_2 respectively.

Definition 2.9. Let \mathcal{A}_1 be an S -ring over G_1 and \mathcal{A}_2 be an S -ring over G_2 . Then the set

$$\mathcal{S} = \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) = \{X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1), X_2 \in \mathcal{S}(\mathcal{A}_2)\}$$

forms a partition of the group $G = G_1 \times G_2$ that defines an S -ring over G . This S -ring is called the *tensor product* of the S -rings \mathcal{A}_1 and \mathcal{A}_2 and denoted by $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Definition 2.10. Let \mathcal{A}_1 be an S -ring over group G_1 with the identity element e_1 and \mathcal{A}_2 be an S -ring over group G_2 with the identity element e_2 . Then the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathcal{S}_1 = \{X_1 \times \{e_2\} : X_1 \in \mathcal{S}(\mathcal{A}_1)\}, \mathcal{S}_2 = \{G_1 \times \{X_2\} : X_2 \in \mathcal{S}(\mathcal{A}_2) \setminus \{e_2\}\}$$

forms a partition of the group $G = G_1 \times G_2$ that defines an S -ring over G . This S -ring is called *the wreath product* of the S -rings \mathcal{A}_1 and \mathcal{A}_2 and denoted by $\mathcal{A}_1 \wr \mathcal{A}_2$.

Lemma 2.13. *The S -ring $\mathcal{A}_1 \otimes \mathcal{A}_2$ is schurian if and only if so are the S -rings \mathcal{A}_1 and \mathcal{A}_2 .*

Lemma 2.14. *The S -ring $\mathcal{A}_1 \wr \mathcal{A}_2$ is schurian if and only if so are the S -rings \mathcal{A}_1 and \mathcal{A}_2 .*

Definition 2.11. Let \mathcal{A} be an S -ring over G and L, U be \mathcal{A} -subgroups such that $e \leq L \leq U \leq G$, $L \trianglelefteq G$. The S -ring \mathcal{A} is called *the generalized wreath product* or *U/L -wreath product* of \mathcal{A}_U and $\mathcal{A}_{G/L}$ if every basic set $S \in \mathcal{S}(\mathcal{A})$ outside U is a union of L -cosets. The product is called *nontrivial* or *proper* if $L \neq 1$, $U \neq G$.

Remark. When $U = L$ the generalized wreath product coincides with a wreath product.

Definition 2.12. Permutation groups $\Gamma, \Gamma_1 \leq \text{Sym}(V)$ are called *2-equivalent* if $\text{Orb}(\Gamma, V^2) = \text{Orb}(\Gamma_1, V^2)$.

Definition 2.13. A permutation group $\Gamma \leq \text{Sym}(V)$ is called *2-isolated* if it is the only group which is 2-equivalent to Γ .

Theorem 2.4. *Let \mathcal{A} be an S -ring over an abelian group G . Suppose that \mathcal{A} is an U/L -wreath product and S -rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ are schurian. Then \mathcal{A} is schurian if and only if there exist groups $\Delta_0 \geq (G/L)_{\text{right}}$ and $\Delta_1 \geq U_{\text{right}}$ such that Δ_0 is 2-equivalent to $\text{Aut}(\mathcal{A}_{G/L})$, Δ_1 is 2-equivalent to $\text{Aut}(\mathcal{A}_U)$ and $(\Delta_0)^{U/L} = (\Delta_1)^{U/L}$.*

Corollary 2.1. *Under the hypothesis of Theorem 2.4 the S -ring \mathcal{A} is schurian whenever the group $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated.*

Lemma 2.15. *Let \mathcal{A} be an S -ring over G . Suppose that the point stabilizer of $\text{Aut}(\mathcal{A})$ has a faithful regular orbit. Then $\text{Aut}(\mathcal{A})$ is 2-isolated.*

Definition 2.14. An S -ring \mathcal{A} is called *quasi-thin* if $|X| \leq 2$ for every $X \in \mathcal{S}(\mathcal{A})$.

Definition 2.15. A basic set $X \neq \{e\}$ of a quasi-thin S -ring \mathcal{A} is called *an orthogonal*, if $X \subseteq YY^{-1}$ for some $Y \in \mathcal{S}(\mathcal{A})$.

Lemma 2.16. *Any commutative quasi-thin S -ring \mathcal{A} is schurian. Moreover, if it has at least two orthogonals, then the group $\text{Aut}(\mathcal{A})_e$ has a faithful regular orbit.*

3. S -RINGS OVER $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$: BASIC SETS CONTAINING ELEMENTS OF ORDER 3

A set $X \subset D$ is called *highest* (in D) if it contains an element of order 3^n . Given an S -ring \mathcal{A} over D set $\text{rad}(\mathcal{A})$ to be the group generated by the groups $\text{rad}(X)$, where X runs over the highest basic sets of \mathcal{A} . Clearly, $\text{rad}(\mathcal{A}) = e$ if and only if every highest basic set of \mathcal{A} has trivial radical. A set $X \subset D$ is called *regular* if it consists of elements of the same order. An S -ring \mathcal{A} over a group D is called *regular* if each highest basic set of \mathcal{A} is regular. A set $X \subset D$ is called *rational* if $X = \cup_m X^{(m)}$, where m runs over integers coprime to 3. An S -ring \mathcal{A} over a group D is called *rational* if each basic set of \mathcal{A} is rational.

In this section we describe the basic sets containing an element of order 3. The main result of this section is given by the following two lemmas.

Lemma 3.1. *Let \mathcal{A} be an S -ring over D . Then exactly one of the following statements holds:*

- (1) T_{C_1} is rational, nonregular, and $T_{C_1} \cup L$ is an \mathcal{A} -subgroup for some \mathcal{A} -subgroup $L \leq E$ such that $L \cap C_1 = \{e\}$;
- (2) T_{C_1} is regular. In this case C_1 is an \mathcal{A} -subgroup or E is an \mathcal{A} -subgroup.

Lemma 3.2. *Let \mathcal{A} be an S -ring over D and T_{c_1} is rational and nonregular. Denote the set of all basic sets that contain an element of order 3 by I . Then exactly one of the following statements holds:*

- (1) $I = \{T_{c_1}, \{q, q^2\}, \{q, q^2\}T_{c_1}\}$, where $q \in E \setminus C_1$ and $T_{c_1} \cup \{e\}$ is a cyclic \mathcal{A} -subgroup;
- (2) $I = \{T_{c_1}, \{q, q^2\}\}$, where $q \in E \setminus C_1$ and $T_{c_1} \cup \{e, q, q^2\}$ is an \mathcal{A} -subgroup;
- (3) $I = \{T_{c_1}\}$;
- (4) $I = \{T_{c_1}, \{q\}, \{q^2\}, qT_{c_1}, q^2T_{c_1}\}$, where $q \in E \setminus C_1$ and $T_{c_1} \cup \{e\}$ is a cyclic \mathcal{A} -subgroup;
- (5) $C_1 \leq \text{rad}(T)$ for every $T \in I \setminus \{T_{c_1}\}$.

The proofs of Lemma 3.1 and Lemma 3.2 will be given later. Now we need to prove an auxiliary lemma.

Lemma 3.3. *Let T_{c_1} be nonregular. Then T_{c_1} is rational.*

Proof. Assume the contrary. Let $T_{c_1} \cap E = Y$, $|T_{c_1}| = a$, and $\alpha = |Y| - |c_1 Y \cap Y|$. Note that $\alpha \geq 1$ because $c_1 \in Y$ but $c_1 \notin c_1 Y$. Since T_{c_1} is not rational, $|Y| \leq 4$ and if $|Y| = 4$ then $|c_1 Y \cap Y| \geq 1$. Therefore $\alpha \leq 3$. Lemma 2.10 implies that $C_1 \leq \text{rad}(T_{c_1}) \setminus Y$. So

$$C_{T_{c_1} T_{c_1}^{-1}}^{T_{c_1}} = |c_1 T_{c_1} \cap T_{c_1}| = a - \alpha.$$

Hence every element from $T_{c_1} \cup T_{c_1}^2$ enters the element $\underline{T_{c_1}} \underline{T_{c_1}}^{-1}$ with coefficient $a - \alpha$. Since e enters $\underline{T_{c_1}} \underline{T_{c_1}}^{-1}$ with coefficient a , we conclude that at least $2a^2 - 2\alpha a + a$ elements enter $\underline{T_{c_1}} \underline{T_{c_1}}^{-1}$. On the other hand, exactly a^2 elements enter this element. Thus we arrive to a contradiction if $2a^2 - 2\alpha a + a > a^2$ or equivalently $a > 2\alpha - 1$. However, this holds for $\alpha = 1$ because T_{c_1} is nonregular; if $\alpha = 2$ then $a > 4$ by Lemma 2.10; if $\alpha = 3$ then $a > 5$ by Lemma 2.10. \square

Proof of Lemma 3.1. If T_{c_1} is not rational then T_{c_1} is regular by Lemma 3.3. In this case $C_1 = \langle T_{c_1} \rangle$ or $E = \langle T_{c_1} \rangle$ and Lemma 2.6 implies Statement (2) of the lemma. If T_{c_1} is regular and rational, obviously, Statement (2) also holds. Suppose that T_{c_1} is rational and nonregular. Then it follows from Lemma 2.10 that $c_1, c_1^2 \in \text{rad}(T_{c_1} \setminus E)$. The set $T_{c_1} \cap E$ is a union of 1, 2, 3 or 4 subgroups of order 3 without $\{e\}$. So there are four possibilities for $T_{c_1} \cap E$:

- (1) $T_{c_1} \cap E = \{c_1, c_1^2\}$;
- (2) $T_{c_1} \cap E = \{c_1, c_1^2, q, q^2\}$ where $|q| = 3$;
- (3) $T_{c_1} \cap E = \{c_1, c_1^2, q, q^2, f, f^2\}$ where $|q| = |f| = 3$;
- (4) $T_{c_1} \cap E = \{c_1, c_1^2, s, s^2, sc_1, s^2 c_1, sc_1^2, s^2 c_1^2\}$.

If $T_{c_1} = \langle T_{c_1} \rangle \setminus \text{rad}(T_{c_1})$ and $\text{rad}(T_{c_1}) = \{e\}$ or $|\text{rad}(T_{c_1})| = 3$ then Statement (1) of the lemma holds with $L = \text{rad}(T_{c_1})$. Let us prove that in all cases $T_{c_1} = \langle T_{c_1} \rangle \setminus \text{rad}(T_{c_1})$ and $\text{rad}(T_{c_1}) = \{e\}$ or $|\text{rad}(T_{c_1})| = 3$. Note that $T_{c_1} \cap C_1 \neq \emptyset$ and $T_{c_1} \setminus C_1 \neq \emptyset$. In the first and fourth cases Lemma 2.10 implies that $C_1 \leq \text{rad}(T \setminus C_1)$. By Theorem 2.1, we have $T_{c_1} = \langle T_{c_1} \rangle \setminus \text{rad}(T_{c_1})$. In the first case $\text{rad}(T_{c_1})$ does not contain c_1 . If $\text{rad}(T_{c_1})$ contains an element of order $m > 3$ then $x^{\frac{m}{3}} = c_1 \in \text{rad}(T_{c_1})$ or $x^{\frac{2m}{3}} = c_1 \in \text{rad}(T_{c_1})$. Therefore $\text{rad}(T_{c_1}) = \{e\}$ or $|\text{rad}(T_{c_1})| = 3$. In the fourth case $\text{rad}(T_{c_1}) = \{e\}$. In the second case

$$|c_1 T_{c_1} \cap T_{c_1}| = |T_{c_1}| - 3 = C_{T_{c_1} T_{c_1}}^{T_{c_1}} = |q T_{c_1} \cap T_{c_1}|. \quad (1)$$

Note that $q\{c_1, c_1^2, q, q^2\} \cap T_{c_1} = \{q^2\}$. Therefore $|q(T_{c_1} \setminus E) \cap T_{c_1}| = |T_{c_1}| - 4 = |T_{c_1} \setminus E|$. We conclude that $q, q^2 \in \text{rad}(T_{c_1} \setminus E)$. Also $c_1, c_1^2 \in \text{rad}(T_{c_1} \setminus E)$ by Lemma 2.10 and $T_{c_1} \cap E \neq \emptyset$, $T_{c_1} \setminus E \neq \emptyset$. Thus, by Theorem 2.1 for separating group E , we have $T_{c_1} = \langle T_{c_1} \rangle \setminus \text{rad}(T_{c_1})$. Since $\text{rad}(T_{c_1})$ does not contain c_1 , the group $\text{rad}(T_{c_1})$ is trivial or has order 3. In the third case (1) holds and $|q\{c_1, c_1^2, q, q^2, f, f^2\} \cap T_{c_1}| = 3$ because $\{f, f^2\} = \{qc_1, q^2 c_1^2\}$ or $\{f, f^2\} = \{q^2 c_1, qc_1^2\}$. So $|q(T \setminus E) \cap T| = |T| - 6 = |T \setminus E|$ and we have $q, q^2 \in \text{rad}(T \setminus E)$. By Theorem 2.1 for separating group E , we have $T_{c_1} = \langle T_{c_1} \rangle \setminus \text{rad}(T_{c_1})$. Since $\text{rad}(T_{c_1})$ does not contain c_1 , it is trivial or has order 3. \square

Lemma 3.4. *In the conditions of Lemma 3.2 let all basic sets from $I \setminus \{T_{c_1}\}$ are rational with trivial radical. Then one of Statements (1) – (3) of Lemma 3.2 holds.*

Proof. Statement (2) of Lemma 3.2 obviously holds if $T_{c_1} \cup \{e, q, q^2\} \leq D$, where $q, q^2 \in E \setminus C_1$. Statement (3) holds if $T_{c_1} \cup \{e\} \leq D$ and $T_{c_1} \cup \{e\}$ is noncyclic. Put $K = T_{c_1} \cup \{e\}$. By Lemma 3.1, we may assume that K is a cyclic group and $K \leq C$.

If $X \in I \setminus \{T_{c_1}\}$ then $|X \cap E| \in \{2, 4, 6\}$. If $|X \cap E| = 6$ then $C_1 \leq \text{rad}(T)$, a contradiction with the assumption of the lemma. So $I \setminus \{T_{c_1}\} = \{X, Y, Z\}$, where $|X \cap E| = |Y \cap E| = |Z \cap E| = 2$, or $I \setminus \{T_{c_1}\} = \{X, Y\}$, where $|X \cap E| = 2$, $|Y \cap E| = 4$. In both cases there exists $X \in I \setminus \{T_{c_1}\}$ such that $X \cap E = \{q, q^2\}$. Note that $|c_1 X \cap X| = |X| - 2$. It is obvious if X is regular and follows from Lemma 2.10 otherwise. Therefore Lemma 2.3 implies that

$$(|X| - 2)|T_{c_1}| = C_{XX}^{T_{c_1}}|T_{c_1}| = C_{T_{c_1}X}^X|X|. \quad (2)$$

Since $(|X|, |X| - 2) \leq 2$, we conclude that $|X| = 2$ or $|T_{c_1}| = \frac{l}{2}|X|$, where $l \geq 1$ is an integer. Moreover, $l \neq 1$ because $|T_{c_1}| \equiv |X| \equiv 2 \pmod{3}$. Every element from T_{c_1} enters the element \underline{X}^2 with coefficient $|X| - 2$ because $|c_1 X \cap X| = |X| - 2$; every element from X enters \underline{X}^2 because $|qX \cap X| \geq 1$; the identity element e enters \underline{X}^2 with coefficient $|X|$ because $X = X^{-1}$. Assuming that $|X| > 2$ and $|T_{c_1}| > \frac{3}{2}|X|$, we conclude that at least $|X|^2 - \frac{3}{2}|X|$ elements of D (counted with multiplicities) enter \underline{X}^2 . However, exactly $|X|^2$ elements enter \underline{X}^2 and $|X|^2 < \frac{3}{2}|X|^2 - |X|$ if $|X| > 2$, a contradiction. Hence $|X| = 2$ or $|T_{c_1}| = |X|$, and

$$\underline{X}^2 = |X|e + (|X| - 2)\underline{T_{c_1}} + \underline{X}. \quad (3)$$

If for $Y \in I \setminus \{T_{c_1}\}$ we have $Y \cap E = \{q, q^2, f, f^2\}$ then $|c_1 Y \cap Y| = |Y| - 2$. Therefore Lemma 2.3 implies that (2) holds for Y . Thus $|T_{c_1}| = \frac{l}{2}|Y|$, where $l \geq 1$ is an integer. Note that $l < 3$ because otherwise at least $\frac{3}{2}|Y|^2 - |Y|$ elements enter the element \underline{Y}^2 that is impossible. Moreover, $l \neq 2$, since $|T_{c_1}| \equiv 2 \pmod{3}$, $|Y| \equiv 1 \pmod{3}$. Thus $|Y| = 2|T_{c_1}|$.

Let $3^k = \max|t|$, $t \in T_{c_1}$, $D_k = \{g \in D : |g| \leq 3^k\}$. Suppose that X contains l elements x_1, \dots, x_l of order greater than 3^k . Then at least $2l$ elements $qx_1, \dots, qx_l, q^2x_1, \dots, q^2x_l$ of order greater than 3^k enter the element $\underline{X} \underline{X}$. On the other hand, due to (3) exactly l elements, namely x_1, \dots, x_l of order greater than 3^k enter the element $\underline{X} \underline{X}$, a contradiction. Thus X does not contain elements of order greater than 3^k and

$$T_{c_1} \cup X \cup T_{c_1}X = D_k \setminus \{e\}. \quad (4)$$

Suppose that $I \setminus \{T_{c_1}\} = \{X, Y, Z\}$ with $|X \cap E| = |Y \cap E| = |Z \cap E| = 2$ and $|X| > 2$, $|Y| > 2$, $|Z| > 2$. Then $|X| = |Y| = |Z| = |T_{c_1}| = m$. Therefore $|c_1 X \cap X| = |qX \cap T_{c_1}| = m - 2$ and $\underline{T_{c_1}}\underline{X} = (m - 2)\underline{X} + \underline{Y} + \underline{Z}$. Due to (4), we have $D_k \setminus \{e\} = T_{c_1} \cup X \cup Y \cup Z$. However, $3m + 2 = |D_k \setminus \{e\}| = |T_{c_1}| + |X| + |Y| + |Z| = 4m$. Since $m > 2$, we have a contradiction. Thus in this case at least one of the sets X, Y, Z has cardinality 2.

Suppose that $I \setminus \{T_{c_1}\} = \{X, Y\}$ with $|X \cap E| = 2$, $|Y \cap E| = 4$ and $|X| > 2$. Then $|X| = |T_{c_1}| = m$, $|Y| = 2|T_{c_1}| = 2m$, and $\underline{T_{c_1}}\underline{X} = (m - 2)\underline{X} + \underline{Y}$. Again (4) implies that $D_k \setminus \{e\} = T_{c_1} \cup X \cup Y \cup Z$ and we have a contradiction. Therefore in all cases there exists a basic set $X \in I \setminus \{T_{c_1}\}$ that has form $\{q, q^2\}$, $q \in E$.

Let us show that $\{q, q^2\}T_{c_1}$ is a basic set. If the elements $qc_1, q^2c_1^2, q^2c_1, qc_1^2$ lie in one basic set Y then $|Y| = 2|T_{c_1}|$ and $Y = \{q, q^2\}T_{c_1}$. Otherwise there are two basic sets

$$Y = qA \cup q^2(T_{c_1} \setminus A), \quad Z = q(T_{c_1} \setminus A) \cup q^2A,$$

where $A \subseteq T_{c_1}$, $T_{c_1} \setminus A = A^{-1}$. Then Lemma 2.10 implies that $c_1 \in \text{rad}(A \setminus C_1)$, $c_1 \in \text{rad}(A^{-1} \setminus C_1)$. Suppose that $yz = c_1$, $y, z \in A$ and $|y| = |z| > 3$. Then $z = c_1y^{-1} \in A^{-1}$, a contradiction. Thus the element c_1 enters the element $\underline{Y} \underline{Z} = (q + q^2)\underline{A} \underline{A}^{-1} + \underline{A}^2 + (\underline{A}^{-1})^2$ with coefficient 1. In the other hand, there exist elements from T_{c_1} that enter the element $\underline{Y} \underline{Z}$ with coefficient at least 2, a contradiction. Therefore $X = \{q, q^2\}T_{c_1}$ is a basic set of \mathcal{A} and Statement (1) of Lemma 3.2 holds. \square

Proof of Lemma 3.2. If $\text{rad}(T) > e$ for some $T \in I \setminus \{T_{c_1}\}$ then $C_1 \leq \text{rad}(T)$ for every $T \in I \setminus \{T_{c_1}\}$ and Statement (5) holds. Thus due to Lemma 3.4 we may assume that there exists nonrational

$T \in I \setminus \{T_{c_1}\}$ such that $\text{rad}(T) = \{e\}$. Then T contains one, two or three elements of order 3. Suppose that T contains exactly one element q of order 3. Then $C_{TT^{-1}}^{T_{c_1}} = |c_1 T \cap T| = |T| - 1$. This is obvious if T is regular and follows from Lemma 2.10 otherwise. Lemma 2.3 yields that

$$|T_{c_1}|(|T| - 1) = C_{TT^{-1}}^{T_{c_1}}|T_{c_1}| = C_{T_{c_1}T}^T|T|.$$

Thus $|T|$ divides $(|T| - 1)|T_{c_1}|$. If $|T_{c_1}| = l|T|$ where $l > 1$, then at least $|T| + l|T|(|T| - 1)$ elements enter the element $\underline{T} \underline{T}^{-1}$. We have a contradiction because $|T| + l|T|(|T| - 1) > |T|^2$. So either $|T| = 1$ or $|T_{c_1}| = |T|$. In the first case $T = \{q\}$ and Statement (4) of the lemma holds.

In the second case we have that $|T_{c_1}| = |T| \equiv 1 \pmod{3}$ by Lemma 2.10. It follows that T_{c_1} contains four elements $c_1, c_1^2, qc_1, q^2c_1^2$ of order 3. Suppose that $c_1 = tx$, $t \in T_{c_1}$, $x \in T$. If $|t| > |x| \geq 3$ then $|tx| = |t| > 3$. The same is true if $|x| > |t| \geq 3$. So $|x| = |t|$. If $|t| > 3$ then $x = c_1 t^{-1} \in T_{c_1}$ by Lemma 2.10. Thus $|x| = |t| = 3$. Then c_1^2 enters the element $\underline{T} \underline{T}_{c_1}$ whereas c_1 does not, a contradiction with rationality of T_{c_1} .

Suppose that T contains exactly three elements of order 3. We consider the case when $s, sc_1, s^2c_1 \in T$, $s^2, s^2c_1^2, sc_1^2 \in T^{-1}$. In other cases such that $|T \cap E| = 3$ the arguments are similar. Note that $|c_1 T \cap T| = |T| - 2$ and $(|T| - 2)|T_{c_1}|$ is divisible by $|T|$. Lemma 2.10 implies that $|T|$ is divisible by 3. However, $(|T| - 2)|T_{c_1}|$ is not divisible by 3 because T_{c_1} contains two elements of order 3, a contradiction.

Suppose that T contains exactly two elements of order 3.

Case 1. Let $s, s^2c_1^2 \in T$, $s^2, sc_1 \in T^{-1}$. Denote the basic set containing sc_1^2 by X . Note that c_1 and c_1^2 appear in $\underline{X} \underline{T}$ only as the product of two elements of order 3 because otherwise $X \cap T^{-1} \neq \emptyset$ by Lemma 2.10. So c_1 enters the element $\underline{X} \underline{T}$ whereas c_1^2 does not, a contradiction with $T_{c_1} = T_{c_1^2}$.

Case 2. Let $s, sc_1 \in T$, $s^2, s^2c_1^2 \in T^{-1}$. Then $s^2c_1 \in X$ since otherwise X contains exactly one element of order 3, hence, $|X| = 1$, and Statement (4) of the lemma holds. The element sc_1^2 enters the element $\underline{T}_{c_1} \underline{T}$ as the product of s and c_1^2 . Since $sc_1^2, s^2c_1 \in X$, the element s^2c_1 also enters $\underline{T}_{c_1} \underline{T}$. Thus there are elements $r \in T_{c_1}$ and $t \in T$ such that $|r| > 3$, $|t| > 3$, and $rt = c_1 s^2$. From Lemma 2.10 it follows that $r^{-1}c_1 \in T_{c_1}$. By Lemma 3.1, either $T_{c_1} \cup \{e\} \leq D$ or $T_{c_1} \cup \{e, q, q^2\} \leq D$, $|q| = 3$. The latter case is impossible because then $T_{c_1} = T$. So $K = T_{c_1} \cup \{e\}$ is an \mathcal{A} -subgroup. We have that $|sK \cap T| \equiv 1 \pmod{3}$ and $|s^2K \cap T| \equiv 0 \pmod{3}$. Thus Lemma 2.5 yields that $s^2K \cap T = \emptyset$. Otherwise $t = r^{-1}c_1 s^2 \in s^2K \cap T$, a contradiction.

Other cases such that $|T \cap E| = 2$ are similar to Case 1 or to Case 2. We conclude that T does not contain exactly two elements of order 3. \square

Corollary 3.1. *Let $3^k = \max|t|$, $t \in T_{c_1}$, $D_k = \{g \in D : |g| \leq 3^k\}$, T_{c_1} be nonregular, rational, and R be the union of all basic sets containing elements of order 3. Then $(D_k \setminus \{e\}) \subseteq R$. Moreover, if $\text{rad}(T) = \{e\}$ for every $T \in I \setminus \{T_{c_1}\}$ then $R = D_k \setminus \{e\}$.*

Proof. The statement of the corollary is clear if one of Statements (1)–(4) of Lemma 3.2 holds. Suppose that Statement (5) of Lemma 3.2 holds. Then $K = T_{c_1} \cup \{e\}$ is a cyclic group. Let $T \in I \setminus \{T_{c_1}\}$ and $q \in T \cap E$. Since $\text{rad}(T)$ is an \mathcal{A} -subgroup, we conclude that $T_{c_1} = K \setminus \{e\} \subseteq \text{rad}(T)$. Thus $qK \subseteq T$, $q^2K \subseteq T^{-1}$, and $R = T_{c_1} \cup T \cup T^{-1} \supseteq T_{c_1} \cup sK \cup s^2K = D_k \setminus \{e\}$. \square

4. S -RINGS OVER $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$: NONREGULAR CASE

The purpose of this section is to describe nonregular S -rings with trivial radical over D . The main result can be formulated as follows.

Theorem 4.1. *Let \mathcal{A} be an S -ring over D . Suppose that $\text{rad}(\mathcal{A}) = e$. Then one of the following statements holds:*

- (1) \mathcal{A} is regular;
- (2) $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_L$, where $\text{rk}(\mathcal{A}_H) = 2$ and $|L| \leq 3 \leq |H|$. In particular, \mathcal{A} is schurian.

The proof of Theorem 4.1 will be given in the end of the section.

Lemma 4.1. *Let T be a nonregular basic set of an S -ring \mathcal{A} over D . Suppose that C_1 is an \mathcal{A} -subgroup. Then $C_1 \leq \text{rad}(T)$.*

Proof. Let $3^m = \min|t|$, $t \in T$. Lemma 2.10 implies that $C_1 \leq \text{rad}(T \setminus T_m)$. Suppose that there exists $t \in T_m$ such that $tc_1 \notin T$. Then $X = T_{tc_1}$ is other than T . Let $\pi : D \rightarrow D/C_1$ be the quotient epimorphism. The sets $\pi(T)$ and $\pi(X)$ are basic sets of \mathcal{A}_{D/C_1} and $\pi(t) \in \pi(T) \cap \pi(X)$. Therefore $\pi(T) = \pi(X)$. So there is an element $y \in T \setminus T_m$ such that either $yc_1 \in X$ or $yc_1^2 \in X$. However, $yc_1, yc_1^2 \in T$ by Lemma 2.10, a contradiction. Thus for every $t \in T_m$ we have $tc_1 \in T$ and $C_1 \leq \text{rad}(T)$. \square

The key point of the proof of Theorem 4.1 is the following statement.

Lemma 4.2. *Let T be a nonregular basic set of an S -ring \mathcal{A} over D . Suppose that T does not contain elements of order 3. Then $\text{rad}(T) > e$.*

Proof. Assume the contrary. Let $3^m = \min|t|$, $t \in T$. Then $m > 1$, $c_1 \notin \text{rad}(T_m)$, $T_m \neq \emptyset$, and $T \setminus T_m \neq \emptyset$. From Lemma 2.12 it follows that $|T_m| \leq 6$. By Lemma 4.1, the group C_1 is not an \mathcal{A} -subgroup. Lemma 3.1 yields that either T_{c_1} is rational and nonregular or E is an \mathcal{A} -subgroup. Let us define groups K and M as in the proof of Lemma 2.12:

$$K = \{\sigma_m : m \in \mathbb{Z}_{3^n}^*\}, \sigma_m : x \rightarrow x^m, M = K_{\{T\}}.$$

Proposition 4.1. *In the above notation $|T \setminus T_m| \geq |T_m|$, and the equality holds if and only if*

- (1) T_m is a union of three M -orbits;
- (2) $T \setminus T_m$ is an M -orbit;
- (3) any element in $T \setminus T_m$ is of order 3^{m+1} .

Proof. Note that $M = (\mathbb{Z}_{3^n}^*)_{\{T_m\}}$. For every $t \in T$ we have $K_t \leq M$ since T is a basic set of \mathcal{A} . Therefore

$$M_t = K_t = \{1 + |t|k : k = 0, \dots, \frac{3^n}{|t|} - 1\}.$$

Thus $|M_t| = \frac{3^n}{|t|}$. It follows that if $|t_1| = |t_2|$ then $|t_1 M| = |t_2 M|$. Let $x \in T_m$. Then $|T_m| = |T_m/M||xM|$. Let $z \in T \setminus T_m$, $|z| > |x|$. Then

$$|M_z| = |M_x| \frac{|x|}{|z|}.$$

Taking into account that T_m is a disjoint union of at most three M -orbits, we obtain that $|M_z| \leq \frac{|M_x|}{3}$ and

$$|T| - |T_m| \geq |zM| \geq 3|xM| \geq |T_m/M||xM| = |T_m|.$$

as required. Since the equality holds only if the second and third inequalities in the above formula are equalities, we are done. \square

Let $H = \langle T_{c_1} \rangle$ be a group of exponent 3^k and let R be the union of all basic sets containing elements of order 3. Then R is a rational \mathcal{A} -set and $R \cap T = \emptyset$. Moreover, $k < m$: if T_{c_1} is rational and nonregular then this follows from Corollary 3.1; otherwise $T_{c_1} \subseteq E$ and, hence, $k = 1 < m$. So $|tk| = |t|$ for every $t \in T$ and every $k \in H$. This implies that $tH \cap T \subseteq T_m$ for every $t \in T_m$. Therefore T_m is a disjoint union of some sets $tH \cap T$ with such t . However, by Lemma 2.5, the number $\lambda := |tH \cap T|$ does not depend on the choice of $t \in T$. Thus λ divides $|T_m|$. From Lemma 2.12 it follows that $|T_m| \in \{1, 2, 3, 4, 6\}$ and, hence,

$$\lambda \in \{1, 2, 3, 4, 6\}.$$

Let us prove that T_{c_1} is regular. Assume the contrary. Then we have that $H = T_{c_1} \cup \{e\}$ or $H = T_{c_1} \cup \{e, q, q^2\}$, $q \in E \setminus C_1$ by Lemma 3.1. In the first case $C_{T_{c_1}}^T = \lambda - 1$. In the second

there exists $t \in T$ such that $qt \notin T$ or $q^2t \notin T$; otherwise $tq, tq^2 \in T$ for every $t \in T$ and, hence, $q \in \text{rad}(T) > e$, a contradiction. So $C_{TT_{c_1}}^T = \lambda - 1$ or $C_{TT_{c_1}}^T = \lambda - 2$. Lemma 2.3 yields that

$$C_{TT_{c_1}}^T |T| = C_{TT_{c_1}}^{T_{c_1}} |T_{c_1}| = |c_1 T \cap T| |T_{c_1}| = (|T| - \alpha) |T_{c_1}|, \quad (5)$$

where $\alpha = |T_m| - |c_1 T_m \cap T_m|$. We conclude that $\alpha > 0$ since $c_1 \notin \text{rad}(T_m)$. Proposition 4.1 implies that $|T| - \alpha > 0$. The number $|T_{c_1}| - C_{TT_{c_1}}^T$ is not equal to 0 because otherwise $\alpha = 0$. Therefore from (5) and Proposition 4.1 it follows that

$$\frac{\alpha |T_{c_1}|}{|T_{c_1}| - C_{TT_{c_1}}^T} = |T| \geq 2|T_m| \geq 2\alpha.$$

Thus $|T_{c_1}| \leq 2C_{TT_{c_1}}^T \leq 10$ because $C_{TT_{c_1}}^T \leq \lambda - 1 \leq 5$. So $|H| = 9$ and, hence, $H = E$ or H is a cyclic group. Let us show that the latter case is impossible. If H is a cyclic group then we may assume that $H \leq C$. On the one hand, $|tH \cap T| \geq 3$ for $t \in T \setminus T_m$ by Lemma 2.10. On the other hand, Lemma 2.12 implies that $0 < |tH \cap T| \leq |tC \cap T_m| \leq 2$ for any $t \in T_m$. We have a contradiction with Lemma 2.5. Therefore $H = E$ and T_{c_1} is regular.

From Lemma 2.10 it follows that $tC_1 \subseteq tH \cap T$ for every $t \in T \setminus T_m$. Thus $\lambda = 3$ or $\lambda = 6$. To complete the proof of Lemma 4.2, we show that both of these cases are impossible. Since λ divides $|T_m|$ and $|T_m| \leq 6$, we have $|T_m| = 3$ or $|T_m| = 6$. Besides, Lemma 2.12 yields that $|c_1 T_m \cap T_m| = 0$ (otherwise $C_1 \leq \text{rad}(T)$) and $\alpha = |T_m|$. Suppose that $\lambda = 6$. Then $|tH \cap T_m| = 6$ for any $t \in T_m$. Hence $T_m \subseteq tH$ and $|c_1 T_m \cap T_m| > 0$, a contradiction. So $\lambda = 3$. Then the regularity of T_{c_1} implies that $|T_{c_1}| \in \{2, 3, 4, 6, 8\}$. If $|T_{c_1}| \in \{4, 6, 8\}$ then T_{c_1} is rational and $C_{TT_{c_1}}^T = \lambda - 1 = 2$ because for any $t \in T \setminus T_m$ we have $c_1 t, c_1^2 t \in tT_{c_1} \cap T$ by Lemma 2.10. If $|T_{c_1}| \in \{2, 3\}$ then $C_{TT_{c_1}}^T = \lambda - 2 = 1$ because $t, c_1^2 t \in tH \cap T$ but $t, c_1^2 t \notin tT_{c_1} \cap T$ for any $t \in T \setminus T_m$. On the other hand, from (5) it follows that

$$|T| = \alpha + \frac{2\alpha}{|T_{c_1}| - 2} \leq 2\alpha$$

for $|T_{c_1}| \in \{4, 6, 8\}$ and

$$|T| = \alpha + \frac{\alpha}{|T_{c_1}| - 1} \leq 2\alpha$$

for $|T_{c_1}| \in \{2, 3\}$. We conclude that $|T| = 2\alpha = 2|T_m| \in \{6, 12\}$ since $|T| \geq 2\alpha$ by Proposition 4.1.

Let $\pi : D \rightarrow D/E$ be the quotient epimorphism and $T' = \pi(T)$. Then T' is a nonregular basic set with trivial radical of a circulant S -ring over D/E . Lemma 2.8 implies that $|T'| \geq 4$. On the other hand, $|T'| = \frac{|T|}{\lambda} = \frac{|T|}{3}$ by the definition of λ . Thus $|T| \neq 6$. Therefore $|T| = 12$. Then $|T'| = 4$. Since T' is a nonregular set with trivial radical, Lemma 2.8 implies that $T' = \langle T' \rangle \setminus \text{rad}(T') = \langle T' \rangle \setminus \{e\}$ and $4 = |T'| = 3^i - 1$ for some integer i , a contradiction. \square

Proposition 4.2. *Suppose that there exists a nonregular highest basic set X of an S -ring \mathcal{A} over D such that $\text{rad}(X) = e$. Then Statement (2) of Theorem 4.1 holds. In particular, $\text{rad}(\mathcal{A}) = e$.*

Proof. Since X is nonregular and $\text{rad}(X) = e$, we conclude by Lemma 4.2 that the set $X \cap E$ is not empty. So neither C_1 nor E is an \mathcal{A} -subgroup (the former follows from Lemma 4.1). So we have that T_{c_1} is nonregular and rational by Statement (2) of Lemma 3.1.

Show that T_{c_1} is a highest basic set. This is obvious if $T_{c_1} = X$. If $T_{c_1} \neq X$ then every basic set T with $T \cap E \neq \emptyset$ and $T \neq T_{c_1}$ has trivial radical (otherwise $c_1 \in \text{rad}(X)$). From Corollary 3.1 it follows that $D_k \setminus \{e\} = R$, where $3^k = \max |t|$, $t \in T_{c_1}$, and R is a union of all basic sets containing elements of order 3. Since $X \subseteq R$ and X is a highest set, this implies that $D_k = D$ and we are done.

Lemma 3.1 implies that $H = T_{c_1} \cup \{e\}$ is an \mathcal{A} -subgroup or $H = T_{c_1} \cup \{e, q, q^2\}$ is an \mathcal{A} -subgroup, where $q \in E \setminus C_1$. In the latter case $H = D$ because T_{c_1} is highest. Thus $T_{c_1} = X$ is the unique highest basic set and $q \in \text{rad}(T_{c_1})$ that is impossible. Thus $H = T_{c_1} \cup \{e\}$. Now, if H is noncyclic then

$H = D$ and $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_L$ for $L = \{e\}$. If H is cyclic then we have $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_L$ for L of order 3 by Statements (1) and (4) of Lemma 3.2. \square

Proof of the Theorem 4.1. Suppose that \mathcal{A} is not regular. Then there exists a nonregular highest basic set X and we are done by Proposition 4.2. \square

5. S -RINGS OVER $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$: REGULAR CASE

In this section we classify regular S -rings with trivial radical over D . The main result is given by the following theorem.

Theorem 5.1. *Let \mathcal{A} be a regular S -ring over D . Suppose that $\text{rad}(\mathcal{A}) = e$. Then \mathcal{A} is cyclotomic. Moreover, \mathcal{A} is Cayley isomorphic to $\text{Cyc}(K, D)$, where $K \leq \text{Aut}(D)$ is one of the groups listed in Table 1. In particular, \mathcal{A} is schurian.*

name	generators	size
K_0	$(x, s) \rightarrow (x, s)$	1
K_1	$(x, s) \rightarrow (x, s^2)$	2
K_2	$(x, s) \rightarrow (x^{-1}, s)$	2
K_3	$(x, s) \rightarrow (x^{-1}, s), (x, s) \rightarrow (x, s^2)$	4
K_4	$(x, s) \rightarrow (x^{-1}, s^2)$	2
K_5	$(x, s) \rightarrow (sx^{-1}, s)$	2
K_6	$(x, s) \rightarrow (sx, sc_1)$	3
K_7	$(x, s) \rightarrow (sx, sc_1), (x, s) \rightarrow (x, s^2 c_1)$	6
K_8	$(x, s) \rightarrow (sx, sc_1^2), (x, s) \rightarrow (x^{-1}, sc_1)$	6
K_9	$(x, s) \rightarrow (sx, sc_1^2), (x, s) \rightarrow (x^{-1}, s^2)$	6

Table 1. The groups of cyclotomic rings with trivial radical.

Before we start to prove Theorem 5.1, we formulate a technical lemma on S -rings over $E = S \times C_1$, where $S = \langle s \rangle$, $C_1 = \langle c_1 \rangle$.

Lemma 5.1. *Let \mathcal{A} be an S -ring over E . Suppose that C_1 is an \mathcal{A} -subgroup. Then \mathcal{A} is one (up to Cayley isomorphism) of the following S -rings:*

- (1) $\mathcal{A} = \mathbb{Z}C_1 \wr \mathcal{A}_S$, where $\text{rk}(\mathcal{A}_S) = 2$;
- (2) $\mathcal{A} = \mathbb{Z}E$;
- (3) $\mathcal{A} = \mathbb{Z}C_1 \wr \mathbb{Z}S$;
- (4) $\mathcal{A} = \mathbb{Z}C_1 \otimes \mathcal{A}_S$, where $\text{rk}(\mathcal{A}_S) = 2$;
- (5) $\mathcal{A} = \mathcal{A}_{C_1} \otimes \mathbb{Z}S$, where $\text{rk}(\mathcal{A}_{C_1}) = 2$;
- (6) $\mathcal{A} = \mathcal{A}_{C_1} \wr \mathbb{Z}S$, where $\text{rk}(\mathcal{A}_{C_1}) = 2$;
- (7) $\mathcal{A} = \text{Cyc}(M, E)$, where $M = \{e, \delta\}$, $\delta : x \rightarrow x^{-1}$;
- (8) $\mathcal{A} = \mathcal{A}_{C_1} \otimes \mathcal{A}_S$, where $\text{rk}(\mathcal{A}_Q) = \text{rk}(\mathcal{A}_{C_1}) = 2$;
- (9) $\mathcal{A} = \mathcal{A}_{C_1} \wr \mathcal{A}_S$, where $\text{rk}(\mathcal{A}_S) = \text{rk}(\mathcal{A}_{C_1}) = 2$;

and $\mathcal{S}(\mathcal{A})$ without e is one (up to Cayley isomorphism) of the following forms :

- (1) $\mathcal{S}(\mathcal{A}) = \{\{c_1\}, \{c_1^2\}, \{s, sc_1, sc_1^2, s^2, s^2 c_1, s^2 c_1^2\}\}$;
- (2) $\mathcal{S}(\mathcal{A}) = \{\{c_1\}, \{c_1^2\}, \{s\}, \{s^2\}, \{sc_1\}, \{s^2 c_1\}, \{sc_1^2\}, \{s^2 c_1^2\}\}$;
- (3) $\mathcal{S}(\mathcal{A}) = \{\{c_1\}, \{c_1^2\}, \{s, sc_1, sc_1^2\}, \{s^2, s^2 c_1, s^2 c_1^2\}\}$;
- (4) $\mathcal{S}(\mathcal{A}) = \{\{c_1\}, \{c_1^2\}, \{s, s^2\}, \{sc_1, s^2 c_1\}, \{sc_1^2, s^2 c_1^2\}\}$;
- (5) $\mathcal{S}(\mathcal{A}) = \{\{c_1, c_1^2\}, \{s\}, \{s^2\}, \{sc_1, sc_1^2\}, \{s^2 c_1, s^2 c_1^2\}\}$;
- (6) $\mathcal{S}(\mathcal{A}) = \{\{c_1, c_1^2\}, \{s, sc_1, sc_1^2\}, \{s^2, s^2 c_1, s^2 c_1^2\}\}$;
- (7) $\mathcal{S}(\mathcal{A}) = \{\{c_1, c_1^2\}, \{s, s^2\}, \{sc_1, s^2 c_1\}, \{sc_1^2, s^2 c_1^2\}\}$;
- (8) $\mathcal{S}(\mathcal{A}) = \{\{c_1, c_1^2\}, \{s, s^2\}, \{sc_1, s^2 c_1^2, sc_1^2, s^2 c_1\}\}$;
- (9) $\mathcal{S}(\mathcal{A}) = \{\{c_1, c_1^2\}, \{s, s^2, sc_1, s^2 c_1^2, sc_1^2, s^2 c_1\}\}$.

Proof. Follows from calculations in the group ring of E that made by [5]. \square

Proof of the Theorem 5.1. Let X be a highest basic set and $x \in X$. Without loss of generality we may assume that $\langle x \rangle = C$. From Lemma 2.12 it follows that $|X| \in \{1, 2, 3, 4, 6\}$. Let us consider all these cases.

Case 1: $|X| = 1$. In this case $X = \{x\}$. Lemma 2.6 implies that C is an \mathcal{A} -subgroup and $\mathcal{A}_C = \mathbb{Z}C$. Suppose that the basic set T_s that contains s is not regular. Then from Lemma 4.1 it follows that $C_1 \leq \text{rad}(T_s)$. Hence $|T_s| \geq 6$. Let Y be the highest basic set containing sx^{-1} . Suppose that $|Y| = 6$. Then $Y_{0e} \neq \emptyset$, $Y_{1s} \neq \emptyset$, $Y_{2s^2} \neq \emptyset$, and $Y = Y^{-1}$. Therefore every highest basic set is rationally conjugate to Y that is not true for X . Thus $|Y| \leq 4$. By Lemma 2.4, the set $x^{-1}Y$ is a basic set containing s . So $x^{-1}Y = T_s$. However, $|x^{-1}Y| \leq 4$ and $|T_s| \geq 6$, a contradiction. Therefore T_s is regular. This implies that E is an \mathcal{A} -subgroup. Since also $c_1, c_1^2 \in \mathcal{A}$, we conclude that $\mathcal{S}(\mathcal{A}_E)$ is one (up to Cayley isomorphism) of the forms (1) – (4) from Lemma 5.1.

If $\mathcal{S}(\mathcal{A}_E)$ is of the form (1) then the set $X_1 = x\{s, sc_1, sc_1^2, s^2, s^2c_1, s^2c_1^2\}$ is a highest basic set such that $c_1 \in \text{rad}(X_1)$, a contradiction. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (3) the set $X_1 = x\{s, sc_1, sc_1^2\}$ is a highest basic set such that $c_1 \in \text{rad}(X_1)$, a contradiction. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (2) or (4) then $\mathcal{A}_C = \mathbb{Z}C$ and S is an \mathcal{A} -subgroup. Therefore \mathcal{A} is the tensor product of \mathcal{A}_C and \mathcal{A}_S . In the former case $\mathcal{A} = \mathbb{Z}D = \text{Cyc}(K_0, D)$; in the latter case $\mathcal{A} = \text{Cyc}(K_1, D)$.

Case 2: $|X| = 2$. Let $X = \{x, x_1\}$. If $x_1 \notin C$ then without loss of generality we may assume that $x_1 \in sC$. If $x_1 \in C$ then Lemma 2.12 implies that $x_1 = x^{-1}$. In the first case put $y = s^2x_1$. Note that $y \in C$.

Let $X = \{x\} \cup s\{y\}$. Since C is cyclic, there exists $k \in \mathbb{Z}$ such that $y = x^k$. By Theorem 2.2, the set $X^{(2)} = \{x^2\} \cup s^2\{y^2\}$ is basic. Since $2sxy = \underline{X}^2 - \underline{X}^{(2)}$, we conclude that $Y = \{sxy\} \in \mathcal{S}(\mathcal{A})$. If $k \equiv 1 \pmod{3}$ then Y is a highest basic set of cardinality 1, and we are done by the previous case. So we may assume that $k \equiv 2 \pmod{3}$. Theorem 2.2 yields that the set $Z = X^{(-k)} = \{x^{-k}\} \cup s\{x^{-k^2}\}$ is basic.

Consider the element

$$\xi = \underline{X} \underline{Z} = x^{-k+1} + sx^{-k^2+1} + s + s^2x^{-k^2+k}. \quad (6)$$

The elements x^{-k+1} and $s^2x^{-k^2+k}$ have order 3^n because $k \equiv 2 \pmod{3}$. Hence $T_s = \{s\}$ or $T_s = \{s, sx^{-k^2+1}\}$. In the latter case T_s is regular because otherwise $|T_s| \geq 4$ by Lemma 2.10. If $T_s = \{s, sx^{-k^2+1}\}$ then $2s^2x^{-k^2+1} = \underline{T_s}^2 - \underline{T_s}^{(2)}$ and $\{s^2x^{-k^2+1}\} \in \mathcal{S}(\mathcal{A})$. Therefore there exists a basic set that has form $\{q\}$, $q \in E \setminus C_1$. Without loss of generality we may assume that $T_s = \{s\}$. Then the sets

$$sX = s\{x\} \cup s^2\{y\}, \quad s^2X = s^2\{x\} \cup \{y\}$$

are basic. The latter implies that $X^{(k)} = s^2X$. Thus $x^{k^2} = y^k = x$ and, hence, 3^n divides $k^2 - 1 = (k-1)(k+1)$. So 3^n divides $k+1$. This shows that $y = x^{-1}$. Since C is cyclic, Theorem 2.2 yields that every highest basic set is rationally conjugate to X or to sX . Hence every highest basic set has one of the following three forms

$$\{x\} \cup s\{x^{-1}\}, \quad s\{x\} \cup s^2\{x^{-1}\}, \quad s^2\{x\} \cup \{x^{-1}\}, \quad x \in C, \quad |x| = 3^n.$$

From Theorem 2.3 it follows that $\{u, u^{-1}\}$ is a basic set for every $u \in C$, $|u| < 3^n$. Since $\{s\} \in \mathcal{S}(\mathcal{A})$, the sets $s\{u, u^{-1}\}$, $s^2\{u, u^{-1}\}$ are basic for every $u \in C$, $|u| < 3^n$. Thus we have $\mathcal{A} = \text{Cyc}(K_5, D)$.

Now let $X = \{x, x^{-1}\}$. Then C is an \mathcal{A} -subgroup by Lemma 2.6. The basic sets of \mathcal{A}_C are of the form $\{x^k, x^{-k}\}$, $k \in \mathbb{Z}$, by Theorem 2.2 and Theorem 2.3. Let us show that

$$\text{rad}(T_s) = e. \quad (7)$$

Assume the contrary. Then $sc_1, sc_1^2 \in T_s$. Let Y be the highest basic set containing sx . Note that $|Y| \in \{2, 4\}$ because otherwise Theorem 2.2 implies that every highest basic set has cardinality 3 or 6. So $Y = s\{x\} \cup s^2\{x^{-1}\}$ or $Y = s\{x, y\} \cup s^2\{x^{-1}, y^{-1}\}$, $y \in C$. The element s enters the element $\psi = \underline{X} \underline{Y}$. Therefore sc_1 and sc_1^2 enter ψ . If $|Y| = 2$ then only the elements s and sx^2 from sC enter ψ , a contradiction. If $|Y| = 4$ then only the elements $s, sx^2, sxy, sx^{-1}y$ from sC enter ψ . However, sx^2 and one of the elements $sxy, sx^{-1}y$ have order 3^n , a contradiction. The obtained contradiction proves

(7). Moreover, T_s is regular since otherwise we have $C_1 \leq \text{rad}(T_s)$ by Lemma 4.1. Hence $E = \langle T_{c_1}, T_s \rangle$ is an \mathcal{A} -subgroup by Lemma 2.6 and $\mathcal{S}(\mathcal{A}_E)$ is one (up to Cayley isomorphism) of the forms (5), (7), (8) from Lemma 5.1.

If $\mathcal{S}(\mathcal{A}_E)$ is of the form (5) then $\mathcal{A}_S = \mathbb{Z}S$ and C is an \mathcal{A} -subgroup. Therefore, $\mathcal{A} = \mathcal{A}_C \otimes \mathcal{A}_S$ and $\mathcal{A} = \text{Cyc}(K_2, D)$. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (7) or (8) then the set $\{s, s^2\}$ is basic and, hence, every basic set outside C is of the form either $s\{y\} \cup s^2\{y^{-1}\}$ or $s\{y, y^{-1}\} \cup s^2\{y, y^{-1}\}$, where $y \in C$. Assume that there exist basic sets

$$Y = s\{y\} \cup s^2\{y^{-1}\}, y \in C, Z = s\{z, z^{-1}\} \cup s^2\{z, z^{-1}\}, z \in C.$$

Then the elements sz and s^2z^{-1} enter the element $(zy^{-1} + yz^{-1})\underline{Y}$ whereas the elements sz^{-1} and s^2z do not, a contradiction. So all basic sets outside C are of the form $s\{y\} \cup s^2\{y^{-1}\}$, $y \in C$, or all basic sets outside C are of the form $s\{y, y^{-1}\} \cup s^2\{y, y^{-1}\}$, $y \in C$. Therefore $\mathcal{A} = \text{Cyc}(K_4, D)$ or $\mathcal{A} = \mathcal{A}_C \otimes \mathcal{A}_S = \text{Cyc}(K_3, D)$, where \mathcal{A}_S is an S -ring of rank 2 over S .

Case 3: $|\mathbf{X}| = 4$. Lemma 2.12 implies that $X = \{x, x^{-1}\} \cup s\{y\} \cup s^2\{y^{-1}\}$, $y \in C$, because $x \in C$. Obviously,

$$\underline{X} \underline{X} = x^2 + x^{-2} + s^2y^2 + sy^{-2} + 4e + 2sxy + 2s^2xy^{-1} + 2sx^{-1}y + 2s^2x^{-1}y^{-1}. \quad (8)$$

Since $X^{(2)} \in \mathcal{S}(\mathcal{A})$ and exactly two elements from sxy , s^2xy^{-1} , $sx^{-1}y$, $s^2x^{-1}y^{-1}$ have order 3^n , there exists a highest basic set Y such that $|Y| \leq 2$ and we arrive at Case 1 or 2.

Case 4: $|\mathbf{X}| = 3$. In this case $X = \{x\} \cup s^i\{y\} \cup s^j\{z\}$, $x, y, z \in C$, $i, j \in \{1, 2\}$. Since C is cyclic, there exist $k, l \in \mathbb{Z}$ such that $y = x^k, z = x^l$. Note that $T = \{s^i x^{k+1}, s^j x^{l+1}, s^{i+j} x^{k+l}\}$ is an \mathcal{A} -set because

$$s^i x^{k+1} + s^j x^{l+1} + s^{i+j} x^{k+l} = \underline{X}^2 - \underline{X}^{(2)}.$$

If $k \equiv 2 \pmod{3}$ or $l \equiv 2 \pmod{3}$ then T contains exactly one element of order 3^n and we arrive at Case 1. So we assume that $k \equiv 1 \pmod{3}$ and $l \equiv 1 \pmod{3}$. Suppose that $i = j$. Then $s^i y, s^j z \in \langle s^i x \rangle$. From Lemma 2.12 it follows that $s^j z = (s^i y)^{-1}$ that is not true. Therefore $i \neq j$. Without loss of generality we assume that $i = 1, j = 2$.

Let us show that

$$\underline{C_1}, \underline{E} \in \mathcal{A}. \quad (9)$$

Suppose first that $|xE \cap X| = 3$. Then $X = x\{e, s\varepsilon_1, s^2\varepsilon_2\}$, $\varepsilon_1, \varepsilon_2 \in C_1$. Note that $\varepsilon_2 \neq \varepsilon_1^{-1}$ since otherwise $\text{rad}(X) = C_1 > e$. A straightforward computation shows that

$$\underline{E \setminus C_1} = \underline{X} \underline{X}^{-1} - 3e.$$

So $E \setminus C_1$ is an \mathcal{A} -set. Hence $E = \langle E \setminus C_1 \rangle$ is an \mathcal{A} -subgroup by Lemma 2.6 and $C_1 = E \setminus (E \setminus C_1)$ is an \mathcal{A} -subgroup. If $|xE \cap X| < 3$ then from Theorem 2.3 it follows that $X^{[3]} = \{x^3, y^3, z^3\} \subseteq C$ is an \mathcal{A} -set. Suppose that $\{x^3\}$ is a basic set. Then the set $x^3 X^{(-2)}$ is basic. Hence $x^3 X^{(-2)} = X$ and $|xE \cap X| = 3$, a contradiction. Similarly, neither y^3 nor z^3 is a basic set. Therefore $X^{[3]}$ is a basic set. Theorem 2.8 yields that $X^{[3]} \in \text{Orb}(K, C_{n-1})$ for some $K \leq \text{Aut}(C_{n-1})$. If $|X^{[3]}| = 2$ then $\text{rad}(X^{[3]}) = e$ and, by Lemma 2.11, without loss of generality we have $x^{3k} = y^3 = x^3$, $x^{3l} = z^3 = x^{-3}$. This contradicts the fact that $k \equiv 1 \pmod{3}$ and $l \equiv 1 \pmod{3}$. Therefore $|X^{[3]}| = 3$ and this set has nontrivial radical by Lemma 2.11. Thus $X^{[3]} = x^3 C_1$ and $C_1 = \text{rad}(X^{[3]})$ is an \mathcal{A} -subgroup by Lemma 2.6. To prove that E is an \mathcal{A} -subgroup it is sufficient to verify that T_s is regular. Indeed, if T_s is regular then $E = \langle T_s, C_1 \rangle$ is an \mathcal{A} -subgroup by Lemma 2.6. Assume that T_s is not regular. Then $C_1 \leq \text{rad}(T_s)$ by Lemma 4.1 and, hence, $|T_s| \geq 6$. Since $X^{[3]} = x^3 C_1$ is a basic set, Lemma 2.6 implies that $C_{n-1} = \langle X^{[3]} \rangle$ is an \mathcal{A} -subgroup. Let $\pi : G \rightarrow G/C_1$ be the quotient epimorphism. Then $\{\pi(x^3)\}$ is a highest basic set of $\mathcal{A}_{C_{n-1}/C_1}$ and, hence, $\mathcal{A}_{C_{n-1}/C_1} = \mathbb{Z}(C_{n-1}/C_1)$ by Lemma 2.4. The set $\pi(T_s)$ is not regular. Therefore Lemma 4.1 implies that $\text{rad}(\pi(T_s)) > e$ and $|\pi(T_s)| \geq 6$. So $|T_s| \geq 18$. Exactly 9 elements enter the element $\theta_1 = \underline{X} \underline{Y}$, where Y is the highest basic set containing y^{-1} . On the other hand, s enters θ_1 and we conclude that at least 18 elements enter θ_1 , a contradiction. Thus T_s is regular and E is an \mathcal{A} -subgroup.

Now we prove that

$$\{c_1\} \in \mathcal{S}(\mathcal{A}).$$

Without loss of generality we may assume that $x^{3^{n-1}} = c_1$. Since $k \equiv 1 \pmod{3}$ and $l \equiv 1 \pmod{3}$, we have $y^{3^{n-1}} = z^{3^{n-1}} = c_1$ and the set $X^{(3^{n-1}+1)} = c_1X$ is basic. The element c_1 enters the element $\underline{c_1X} \underline{X}^{-1}$ whereas c_1^2 does not. So c_1 and c_1^2 lie in the different basic sets.

Let us check that there are no \mathcal{A} -subgroups of order 3 distinct from C_1 . Assume the contrary. Without loss of generality let S be an \mathcal{A} -subgroup of order 3 other than C_1 . Let π_1 be the quotient epimorphism from D to D/S . The set $\pi_1(X)$ has trivial radical since otherwise $\text{rad}(\pi_1(X)) = C_1$ and, hence, $\text{rad}(X) = \langle sc_1 \rangle$ or $\text{rad}(X) = \langle s^2c_1 \rangle$. Thus we have $\pi_1(X) \in \text{Orb}(K, C)$ for some $K \leq \text{Aut}(C)$ by Theorem 2.8. Lemma 2.11 implies that $\pi_1(X) = \{u\}$ or $\pi_1(X) = \{u, u^{-1}\}$. The first case is impossible because in this case $S \leq \text{rad}(X)$. In the second case $y = x^k = x^{-1}$ or $z = x^l = x^{-1}$. This contradicts the fact that $k \equiv 1 \pmod{3}$ and $l \equiv 1 \pmod{3}$. Thus there are no \mathcal{A} -subgroups of order 3 distinct from C_1 and (9) yields that $\mathcal{S}(\mathcal{A}_E)$ has (up to Cayley isomorphism) form (1) or (3) from Lemma 5.1.

Let Y and Z be the basic sets containing the elements y and z respectively. Then by the above remarks, $Y^{-1} = \{y^{-1}\} \cup s\{u\} \cup s^2\{v\}$, $u, v \in C$, and $Z^{-1} = \{z^{-1}\} \cup s\{u_1\} \cup s^2\{v_1\}$, $u_1, v_1 \in C$. The element s enters the element $\underline{X} \underline{Y}^{-1}$. So the elements sc_1 and sc_1^2 enter $\underline{X} \underline{Y}^{-1}$. Therefore without loss of generality $u = c_1x^{-1}$, $v = c_1^2z^{-1}$. The element s^2 enters the element $\underline{X} \underline{Z}^{-1}$. So the elements s^2c_1 and $s^2c_1^2$ enter $\underline{X} \underline{Z}^{-1}$. Therefore without loss of generality $u_1 = c_1y^{-1}$, $v_1 = c_1^2x^{-1}$. Thus

$$Y = \{y\} \cup s\{c_1z\} \cup s^2\{c_1^2x\}, \quad Z = \{z\} \cup s\{c_1x\} \cup s^2\{c_1^2y\}.$$

By Theorem 2.2, we have $Y = X^{(k)}$, $Z = X^{(l)}$. Since $k \equiv 1 \pmod{3}$ and $l \equiv 1 \pmod{3}$, we conclude that $y^k = c_1z$, $z^k = c_1^2x$, $y^l = c_1x$, $z^l = c_1^2y$. These equalities imply that $x^{k^3} = y^{k^2} = x$, $x^{l^3} = z^{l^2} = x$. Thus 3^n divides $k^3 - 1$ and $l^3 - 1$. Let $k = 3^ap + 1$, where 3 does not divide p and $a \geq 1$. Then

$$k^3 - 1 = 3^{a+1}p(3^{2a-1}p^2 + 3^ap + 1).$$

We have $a + 1 \geq n$ because 3 does not divide $(3^{2a-1}p^2 + 3^ap + 1)$. Hence $y = x^k = q_1x$ where $q_1 \in C_1$. Similarly $z = q_2x$, $q_2 \in C_1$. Since $\text{rad}(X) = e$, we have $q_2 \neq q_1^{-1}$. Every highest basic set is rationally conjugate to X because $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. This implies that if X is an orbit of $K \leq \text{Aut}(D)$ then every highest basic set is an orbit of K .

A straightforward computation shows that $q_1q_2x^3$ is the unique element which enters the element $\underline{X} \underline{X}^{(3)}$ with coefficient 3. So the set $\{x^3\}$ is basic. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (3) from Lemma 5.1 then the basic sets of $\mathcal{A}_{D_{n-1}}$ are of the forms

$$\{u\}, \quad suC_1, \quad s^2uC_1, \quad u \in C_{n-1}$$

by Lemma 2.4, and, hence, \mathcal{A} is Cayley isomorphic to $Cyc(K_6, D)$; if $\mathcal{S}(\mathcal{A}_E)$ is of the form (1) from Lemma 5.1 then the basic sets of $\mathcal{A}_{D_{n-1}}$ are of the forms

$$\{u\}, \quad suC_1 \cup s^2uC_1, \quad u \in C_{n-1}$$

by Lemma 2.4, and, hence, \mathcal{A} is Cayley isomorphic to $Cyc(K_7, D)$.

Case 5: $|X| = 6$. From Lemma 2.12 it follows that

$$X = \{x, x^{-1}\} \cup s\{y, z\} \cup s^2\{y^{-1}, z^{-1}\}, \quad x, y, z \in C.$$

Theorem 2.2 implies that for every highest basic Y there exists $m \in \mathbb{Z}$ such that $Y = X^{(m)}$. Since C is cyclic, there exist $k, l \in \mathbb{Z}$ such that $y = x^k, z = x^l$. Without loss of generality we may assume that

$$k \equiv 1 \pmod{3}, \quad l \equiv 2 \pmod{3}.$$

Indeed, suppose that $k \equiv l \pmod{3}$. Then the element s^2yz has order 3^n and it enters the element $\theta_2 = \underline{X}^2 - \underline{X}^{(2)}$. So there exists $b \in C$, $|b| = 3^n$ that enters θ_2 because every highest basic set nontrivially intersects with C . However, only the elements yz^{-1} , $y^{-1}z$, e from C enter θ_2 . All these elements have order less than 3^n , a contradiction. Therefore $k \not\equiv l \pmod{3}$.

Let us prove (9). The proof is similar to the case when $|X| = 3$. If $|xE \cap X| = 3$ then

$$X = x\{e, s\varepsilon_1, s^2\varepsilon_2\} \cup x^{-1}\{e, s^2\varepsilon_1^2, s\varepsilon_2^2\}, \quad \varepsilon_1, \varepsilon_2 \in C_1,$$

because $X = X^{-1}$. Note that $\varepsilon_2 \neq \varepsilon_1^{-1}$ since otherwise $\text{rad}(X) > e$. Thus all elements from $E \setminus C_1$ enter \underline{X}^2 and only these elements from D_{n-1} enter \underline{X}^2 . Since \mathcal{A} is regular, D_{n-1} is an \mathcal{A} -subgroup. So $E \setminus C_1$ is an \mathcal{A} -set. Lemma 2.6 implies that $E = \langle E \setminus C_1 \rangle$ is an \mathcal{A} -subgroup and we are done.

If $|xE \cap X| < 3$ then from Theorem 2.3 it follows that $X^{[3]} = \{x^3, y^3, z^3, x^{-3}, y^{-3}, z^{-3}\} \subseteq C$ is an \mathcal{A} -set. All basic sets inside $X^{[3]}$ are conjugate and, hence, have the same radical. Suppose that every basic set inside $X^{[3]}$ has trivial radical. Then from Lemma 2.8 and Lemma 2.11 it follows that $T_{x^3} = \{x^3\}$ or $T_{x^3} = \{x^3, x^{-3}\}$. In the former case $x^3 X^{(2)}$ is a basic set by Lemma 2.4. So $x^3 X^{(2)} = X$ and $x^5 = x$, a contradiction with $|x| > 3$. In the latter case $(x^3 + x^{-3})\underline{X}^{(2)}$ contains 12 elements including x and x^5 . This implies that

$$(x^3 + x^{-3})\underline{X}^{(2)} = \underline{X} + \underline{X}^{(5)}. \quad (10)$$

A straightforward computation shows that $s\{x^3 y^{-2}, x^3 z^{-2}, x^{-3} y^{-2}, x^{-3} z^{-2}\}$ is the set of all elements from sC that the summand in the left-hand side of (10) contains. Therefore

$$\{x^3 y^{-2}, x^3 z^{-2}, x^{-3} y^{-2}, x^{-3} z^{-2}\} = \{y, z, y^{-5}, z^{-5}\}.$$

Since $k \equiv 1 \pmod{3}$, $l \equiv 2 \pmod{3}$, we conclude that $y^3, z^3 \in \{x^3, x^{-3}\}$. This yields that $|xE \cap X| = 3$, in contrast to the assumption. Thus $\text{rad}(X^{[3]}) > e$. Moreover, $\text{rad}(X^{[3]}) = C_1$ because $|X^{[3]}| \leq 6$. From Lemma 2.6 it follows that C_1 is \mathcal{A} -subgroup. If T_s is not regular then like in the case when $|X| = 3$ we have that $|T_s| \geq 18$. This contradicts the fact that exactly 16 elements including s from sC enter \underline{X}^2 . Thus T_s is regular and E is an \mathcal{A} -subgroup.

Now we show that $\{c_1, c_1^2\} \in \mathcal{S}(\mathcal{A})$. Assume the contrary. Then $\{c_1\}$ is a basic set. So $c_1 X$ is a basic set. Without loss of generality we assume that $x^{3^{n-1}} = c_1$. Since $k \equiv 1 \pmod{3}$ and $l \equiv 2 \pmod{3}$, we conclude that $y^{3^{n-1}} = c_1$ and $z^{3^{n-1}} = c_1^2$. On the one hand,

$$X^{(3^{n-1}+1)} = \{c_1 x, c_1^2 x^{-1}\} \cup s\{c_1 y, c_1^2 z\} \cup s^2\{c_1^2 y^{-1}, c_1 z^{-1}\}$$

is a basic set containing $c_1 x$. On the other hand, $X^{(3^{n-1}+1)} \neq c_1 X$ because $sc_1 z \in c_1 X$, $sc_1 z \notin X^{(3^{n-1}+1)}$, a contradiction. Thus $\{c_1, c_1^2\} \in \mathcal{S}(\mathcal{A})$.

Let us show that there are no \mathcal{A} -subgroups of order 3 distinct from C_1 . Assume the contrary. Without loss of generality let S be an \mathcal{A} -subgroup of order 3 other than C_1 . Let π_1 be the quotient epimorphism from D to D/S . Suppose that $\text{rad}(\pi_1(X)) > e$. Then $\text{rad}(\pi_1(X)) = C_1$ because $|\pi_1(X)| \leq 6$. This implies that $\pi_1(X) = xC_1 \cup x^{-1}C_1$. So

$$X = x\{e, f_1, f_2\} \cup x^{-1}\{e, f_1^{-1}, f_2^{-1}\},$$

where $f_1, f_2 \in \{sc_1, sc_1^2, s^2 c_1, s^2 c_1^2\}$. Assume that $f_2 \neq f_1^{-1}$. Then $f_2 = f f_1$, where $f \in \{s, s^2, c_1, c_1^2\}$. Note that $\{f, f^2\}$ is an \mathcal{A} -set. The element $f_2 x$ enters $(f + f^2)\underline{X}$ whereas x does not. A contradiction holds because x and $f_2 x$ lie in the same basic set X . Therefore $f_2 = f_1^{-1}$ and $\text{rad}(X) = \{e, f_1, f_1^{-1}\}$ that is not true. Thus $\text{rad}(\pi_1(X)) = e$. Hence Lemma 2.8 and Lemma 2.11 yield that $\pi_1(X) = \{u, u^{-1}\}$. Then $S \leq \text{rad}(X)$, a contradiction. So there are no \mathcal{A} -subgroups of order 3 distinct from C_1 . Thus by the previous paragraph, $\mathcal{S}(\mathcal{A}_E)$ has form (6) or (9) from Lemma 5.1.

Further we prove that $y = q_1 x$, $z = q_2 x^{-1}$, $q_1, q_2 \in C_1$. Using the assumption on l and k , list all elements from sC of order less than 3^n that enter $\underline{X}^{-1} \underline{X}^{(k)}$

$$s, sz^k x, sy^k x^{-1}, syz, sz^{-k} y^{-1}, sy^{-k} z^{-1}. \quad (11)$$

and all elements from sC of order less than 3^n that enter $\underline{X}^{-1} \underline{X}^{(l)}$

$$s, sx^{-1} y^{-l}, syz, sz^l y^{-1}, sy^l z^{-1}, sxz^{-l}. \quad (12)$$

Every element from (11) has the form sx^i , where

$$i \in I = \{kl + 1, k^2 - 1, k + l, kl + k, k^2 + l\},$$

because $y = x^k$, $z = x^l$. Since $c_1 \in \text{rad}(T_s)$ and T_s is contained in $\underline{X}^{-1} \underline{X}^{(k)}$, we conclude that sc_1 and sc_1^2 are among the elements in (11). Therefore two numbers from I are divisible by 3^{n-1} . Let us show that 3^{n-1} divides $k - 1$. This is obvious by the assumption on k if $k^2 - 1$ is a multiple of 3^{n-1} .

If $k+l$ and $kl+k$ are divisible by 3^{n-1} then 3^{n-1} divides $l(k-1) = kl+k-k-l$, hence, 3^{n-1} divides $k-1$ by the assumption on l . If $k+l$ and $kl+1$ are multiple of 3^{n-1} then $(k-1)(l-1) = kl+1-k-l$ is a multiple of 3^{n-1} , hence, $k-1$ is a multiple of 3^{n-1} . If $k+l$ and k^2+l are divisible by 3^{n-1} then $k(k-1) = k^2+l-l-k$ is divisible by 3^{n-1} , hence, $k-1$ is divisible by 3^{n-1} . If 3^{n-1} divides $kl+k$ and $kl+1$ then 3^{n-1} divides $k-1 = kl+k-kl-l$. If $kl+k$ and k^2+l are multiple of 3^{n-1} then $(k-1)(k-l) = k^2+l-kl-k$ is a multiple of 3^{n-1} , hence, $k-1$ is a multiple of 3^{n-1} . If $kl+1$ and k^2+l are divisible by 3^{n-1} then without loss of generality $z^k x = x^{kl+1} = c_1$, $y^{-k} z^{-1} = x^{-k^2-l} = c_1^2$. Note that $x^{-k^3+1} = (x^{-k^2-l})^k x^{kl+1} = c_1^2 c_1 = e$. Thus k^3-1 is divisible by 3^n and $k-1$ is divisible by 3^{n-1} . So in all cases $k-1$ is divisible by 3^{n-1} and we have $y = q_1 x$, $q_1 \in C_1$.

Every element from (12) has the form sx^j , where

$$j \in J = \{kl+1, l^2-1, k+l, kl-l, l^2-k\},$$

because $y = x^k$, $z = x^l$. Again sc_1 and sc_1^2 are among the elements in (12) because T_s enters $\underline{X}^{-1} \underline{X}^{(l)}$. This implies that 3^{n-1} divides two elements from J . Applying the similar arguments, we conclude that $l+1$ is divisible by 3^{n-1} and $z = x^{l+1} x^{-1} = q_2 x^{-1}$, where $q_2 \in C_1$. Note $q_1 \neq q_2$ since otherwise $sq_1 \in \text{rad}(X)$. Every highest basic set is rationally conjugate to X because $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. This implies that if X is an orbit of $K \leq \text{Aut}(D)$ then every highest basic set is an orbit of K .

A straightforward computation shows that only the elements $q_1 q_2^2 x^3$, $q_1^2 q_2 x^{-3}$ enter the element $\underline{X}^{(3)}$ with coefficient 3. If $\{q_1 q_2^2 x^3\}$ is a basic set then Lemma 2.4 implies that $\{c_1\}$ is a basic set that is not true. Therefore $\{q_1 q_2^2 x^3, q_1^2 q_2 x^{-3}\}$ is a basic set and every basic set of $\mathcal{A}_{C_{n-1}}$ is of the form $\{u, u^{-1}\}$, $u \in C_{n-1}$, by Theorem 2.2 and Theorem 2.3. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (6) from Lemma 5.1 then a straightforward check shows that every nonhighest basic set outside C is one of two forms

$$\{su, sc_1 u, sc_1^2 u, su^{-1}, sc_1 u^{-1}, sc_1^2 u^{-1}\}, \{s^2 u, s^2 c_1 u, s^2 c_1^2 u, s^2 u^{-1}, s^2 c_1 u^{-1}, s^2 c_1^2 u^{-1}\}, u \in C.$$

So \mathcal{A} is Cayley isomorphic to $\text{Cyc}(K_8, D)$. If $\mathcal{S}(\mathcal{A}_E)$ is of the form (9) from Lemma 5.1 then a straightforward check shows that every nonhighest basic set outside C is of the form

$$\{su, sc_1 u, sc_1^2 u, s^2 u^{-1}, s^2 c_1 u^{-1}, s^2 c_1^2 u^{-1}\}, u \in C.$$

So \mathcal{A} is Cayley isomorphic to $\text{Cyc}(K_9, D)$. □

Corollary 5.1. *Let \mathcal{A} be a regular S -ring over D such that $\text{rad}(\mathcal{A}) = e$. Then C_1 and C_{n-1} are \mathcal{A} -subgroups.*

Corollary 5.2. *Let \mathcal{A} be a regular S -ring over D such that $\text{rad}(\mathcal{A}) = e$. If L is an \mathcal{A} -subgroup of order 3 and $\mathcal{A} = \text{Cyc}(K_i, D)$, $i \in \{0, 1, 2, 3, 4, 5\}$, then the group $\text{Aut}(\mathcal{A}_{D/L})$ is 2-isolated.*

Proof. For $n \leq 3$ the statement of the corollary follows from calculations in the group ring of D that made by [5]. So we assume that $n \geq 4$. By Lemma 2.15, it is sufficient to prove that $\text{Aut}(\mathcal{A}_{D/L})_e$ has a faithful regular orbit. If $L \neq C_1$ then D/L is cyclic and Theorem 5.1 implies that either $\mathcal{A}_{D/L} = \mathbb{Z}(D/L)$ or every basic set of $\mathcal{A}_{D/L}$ is of the form $\{x, x^{-1}\}$, $x \in C/L$. In both cases, obviously, the group $\text{Aut}(\mathcal{A}_{D/L})_e$ has a faithful regular orbit.

Let $L = C_1$ and $\pi : D \rightarrow D/L$ be the quotient epimorphism. If $\mathcal{A} = \text{Cyc}(K_0, D)$ then $|\text{Aut}(\mathcal{A}_{D/L})_e| = 1$ and, obviously, $\text{Aut}(\mathcal{A}_{D/L})_e$ has a faithful regular orbit. If $\mathcal{A} = \text{Cyc}(K_1, D) = \mathbb{Z}C \otimes \mathcal{A}_S$, where \mathcal{A}_S has rank 2, then $\mathcal{A}_{D/L} = \mathbb{Z}C_{n-1} \otimes \mathcal{A}_S$. So $|\text{Aut}(\mathcal{A}_{D/L})_e| = 2$ and $\pi(\{sx, s^2x\})$ is a faithful regular orbit of $\text{Aut}(\mathcal{A}_{D/L})_e$. If $\mathcal{A} = \text{Cyc}(K_i, D)$, $i \in \{2, 4, 5\}$, then $\mathcal{A}_{D/L}$ is a quasi-thin S -ring with at least two orthogonals. Indeed, $\{\pi(y^2), \pi(y^{-2})\}$ and $\{\pi(y^4), \pi(y^{-4})\}$, where y is a generator of C_{n-1} , are orthogonals. These orthogonals are distinct since $n \geq 4$. Thus $\text{Aut}(\mathcal{A}_{D/L})_e$ has a faithful regular orbit by Lemma 2.16. If $\mathcal{A} = \text{Cyc}(K_3, D)$ the basic sets of $\mathcal{A}_{D/L}$ are of the form

$$\{\pi(s), \pi(s^2)\}, \{\pi(y), \pi(y^{-1})\}, \pi(s)\{\pi(y), \pi(y^{-1})\} \cup \pi(s^2)\{\pi(y), \pi(y^{-1})\}, y \in C.$$

Note that $\pi(S)$ and $\pi(C)$ are $\mathcal{A}_{D/L}$ -subgroups, $\mathcal{A}_{D/L} = \mathcal{A}_{C/L} \otimes \mathcal{A}_{S/L}$, and $\text{Aut}(\mathcal{A}_{D/L})_e|_S \cong \text{Aut}(\mathcal{A}_{D/L})_e|_C \cong \mathbb{Z}_2$. Thus we conclude that $\text{Aut}(\mathcal{A}_{D/L})_e \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\pi(s)\{\pi(y), \pi(y^{-1})\} \cup \pi(s^2)\{\pi(y), \pi(y^{-1})\}$ is a faithful regular orbit of $\text{Aut}(\mathcal{A}_{D/L})_e$. □

Proposition 5.1. *Let \mathcal{A} be an S -ring over D . Suppose that for every highest basic set X with trivial radical the following statements hold:*

- (1) X is regular;
- (2) $\langle X \rangle = D$.

Then every highest basic set of \mathcal{A} has trivial radical.

Proof. Lemma 2.12 implies that $|X| \leq 6$. If $|X| = 1$ then Condition (2) of the Proposition does not hold. The statement of the proposition follows from Theorem 2.2 if $|X| \in \{3, 6\}$. In these cases every highest basic set is rationally conjugate to X because $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. Suppose that $|X| = 2$. Then without loss of generality we may assume that $X = \{x\} \cup s\{y\}$, $x, y \in C$. Theorem 2.3 implies that C_1 is an \mathcal{A} -subgroup. From (6) (see proof of Theorem 5.1) it follows that T_s is regular (otherwise $|T_s| \geq 6$ by Lemma 4.1), $|T_s| \leq 2$, and $T_s \neq T_{s^2}$. Thus E is an \mathcal{A} -subgroup and Lemma 5.1 yields that there exists a basic set of the form $\{q\}$, $q \in E \setminus C_1$. From Lemma 2.4 it follows that qX is a basic set. Therefore

$$\bigcup_m (X^{(m)} \cup (qX)^{(m)}) = D \setminus D_{n-1},$$

where m runs over integers coprime to 3. We are done because $\text{rad}(X) = e$.

Let us show that Condition (2) of the Proposition does not hold whenever $|X| = 4$. In this case without loss of generality we may assume that $X = \{x, x^{-1}\} \cup s\{y\} \cup s^2\{y^{-1}\}$, $x, y \in C$. Theorem 2.3 implies that C_1 is an \mathcal{A} -subgroup. Exactly one of the elements $sxy, sx^{-1}y$, say the first one, has order 3^n . If T_{sxy} is not regular then $|T_{sxy}| \geq 6$ by Lemma 4.1 that contradicts (8) (see proof of Theorem 5.1). Indeed, exactly four distinct elements including sxy enter $\underline{X}^2 - \underline{X}^{(2)}$. So T_{sxy} is regular and from (8) it follows that $T_{sxy} = \{sxy\}$ or $T_{sxy} = \{sxy, s^2x^{-1}y^{-1}\}$. In both cases $\langle T_{sxy} \rangle$ is cyclic. \square

6. S -RINGS OVER $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$: NONTRIVIAL RADICAL CASE

In this section we are interested in an S -ring over D that has nontrivial radical and follow [7, Section 9].

Theorem 6.1. *Let \mathcal{A} be an S -ring over D such that $\text{rad}(\mathcal{A}) > e$. Then \mathcal{A} is a proper S -wreath product. Moreover, $|S| = 1$ or $|S| = 3$ or $\text{rad}(\mathcal{A}_U) = e$ and $|L| = 3$, where $S = U/L$.*

Proof. Denote by U the group generated by all basic sets of \mathcal{A} with trivial radicals.

Lemma 6.1. *U is an \mathcal{A} -subgroup and $\text{rad}(\mathcal{A}_U) = e$.*

Proof. The first statement follows from Lemma 2.6. Let us prove that $\text{rad}(\mathcal{A}_U) = e$. Without loss of generality we may assume that $U = D$. Then there exists a highest basic set X such that $\text{rad}(X) = e$. If X is not regular then it contains an element of order 3 by Lemma 4.2 and we have that $\text{rad}(\mathcal{A}) = e$ by Proposition 4.2. Therefore we may assume that every highest basic set with trivial radical is regular. At least one of the sets X_{0e}, X_{1s}, X_{2s^2} is not empty. Without loss of generality we assume that $X_{0e} \neq \emptyset$. If $\langle Z \rangle = D$ for every highest basic set Z with trivial radical then every highest basic set has trivial radical by Proposition 5.1 and we are done. So we may assume that $\langle X \rangle = C$. Then by Lemma 2.11, we have $X = \{x\}$ or $X = \{x, x^{-1}\}$, where $x \in C$. Since $\langle X \rangle = C$, there exists a basic set Y such that $D = \langle X, Y \rangle$ and $\text{rad}(Y) = e$. If Y is not regular then from Lemma 4.1 it follows that $C_1 \leq \text{rad}(Y)$. So Y is regular. Without loss of generality $Y_{1s} \neq \emptyset$ and $y \in Y_{1s}$. Note that $|Y_{1s}| \leq 2$ by Lemma 2.12.

Let us show that $\text{rad}(T_s) = e$, where T_s is a basic set containing s . It is obvious if $T_s = Y$. Suppose that $T_s \neq Y$. Assume on the contrary that $\text{rad}(T_s) > e$. Then $C_1 \leq \text{rad}(T_s)$. If $|Y_{1s}| = 1$ then exactly two elements s and sy^2 from sC enter the element $\alpha = (y + y^{-1})\underline{Y}$, a contradiction. Suppose that $Y_{1s} = \{y, z\}$. Then exactly four elements $s, syz, sy^{-1}z$, and sy^2 from sC enter α . Since $sc_1, sc_1^2 \in T_s$, we conclude that $\{c_1, c_1^2\} \subseteq \{y^2, yz, y^{-1}z\}$. If $y^2 \in C_1$ then $sc_1 \in Y$ or $sc_1^2 \in Y$ and $Y = T_s$, a contradiction. Thus $\{c_1, c_1^2\} = \{yz, y^{-1}z\}$. Hence $z^2 = e$, a contradiction.

From the claim of the previous paragraph it follows that T_s is regular because otherwise $C_1 \leq \text{rad}(T_s)$ by Lemma 4.1. Since C_1 is an \mathcal{A} -subgroup, from Lemma 5.1 it follows that there exists an \mathcal{A} -subgroup

$Q \neq C_1$ of order 3. Every basic set inside C has the form $\{x\}$ or $\{x, x^{-1}\}$. Hence $\{qx, qx^{-1}, q^2x, q^2x^{-1}\}$ is an \mathcal{A} -set for every $x \in C$. Thus every basic set of \mathcal{A} has trivial radical as required. \square

Lemma 6.2. *Let \mathcal{A} be an S -ring over D and H be a minimal \mathcal{A} -subgroup. Then either $H \setminus \{e\} \in \mathcal{S}(\mathcal{A})$ or $|H| = 3$ and $\mathcal{A}_H = \mathbb{Z}H$.*

Proof. If $|H| > 3$ then the primitive S -ring \mathcal{A}_H is of rank 2 by Lemma 2.7. Thus $H \setminus \{e\} \in \mathcal{S}(\mathcal{A})$. Let $|H| = 3$ and $rk(\mathcal{A}_H) > 2$. Then, obviously, $\mathcal{A}_H = \mathbb{Z}H$. \square

Since by the theorem hypothesis $\text{rad}(\mathcal{A}) > e$, Lemma 6.1 implies that $U < D$. In addition, from Lemma 6.2 it follows that U contains every minimal \mathcal{A} -subgroup.

Lemma 6.3. *If there is a unique minimal \mathcal{A} -subgroup or $C_1 \leq \text{rad}(X)$ for every basic set X outside U then the statement of Theorem 6.1 holds.*

Proof. Let L be a unique minimal \mathcal{A} -subgroup. Then $L \leq \text{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{A})_{D \setminus U}$ because Lemma 2.6 implies that $\text{rad}(X)$ is a nontrivial \mathcal{A} -subgroup. So \mathcal{A} is the U/L -wreath product. If $|L| = 3$ then the statement of Theorem 6.1 holds. Suppose that $|L| > 3$. Then from Corollary 5.1 it follows that \mathcal{A}_U is not regular (otherwise C_1 is an \mathcal{A} -subgroup of order 3). Therefore by Theorem 4.1, the S -ring \mathcal{A}_U has rank 2. Hence $U = L$ and we are done.

To complete the proof suppose that there are at least two minimal \mathcal{A} -subgroups and $C_1 \leq \text{rad}(X)$ for every $X \in \mathcal{S}(\mathcal{A})_{D \setminus U}$. Assume that $C_1 \not\leq U$. Then $|U| = 3$ and U is the unique minimal \mathcal{A} -subgroup because U contains every minimal \mathcal{A} -subgroup, a contradiction. So $C_1 \leq U$. Denote by H the minimal \mathcal{A} -subgroup containing C_1 . Then \mathcal{A} is the S -wreath product, where $S = U/H$. If $H = C_1$ then $|H| = 3$ and the statement of Theorem 6.1 holds. If $H > C_1$ then \mathcal{A}_U is not regular by Corollary 5.1. Thus Theorem 4.1 implies that $\mathcal{A}_U = \mathcal{A}_H \otimes \mathcal{A}_L$ where $|L| \leq 3$. So $|S| = |U/H| \in \{1, 3\}$. \square

The union of all basic sets X such that $\text{rad}(X) = e$ or $c_1 \in \text{rad}(X)$ denote by V . Then, obviously, $U \subset V$ and V is an \mathcal{A} -set. By Lemma 6.3, we may assume that $V \neq D$ and there exist at least two minimal \mathcal{A} -subgroups. Let X be a basic set containing an element of order 3. Suppose that $\text{rad}(X) > e$ and $c_1 \notin \text{rad}(X)$. Then $|\text{rad}(X)| = 3$ and $\text{rad}(X)$ is the unique minimal \mathcal{A} -subgroup, a contradiction. So $X \subseteq V$ and $E \subseteq V$.

For given \mathcal{A} -set X put $\mathcal{S}(\mathcal{A})_X = \{Y \in \mathcal{S}(\mathcal{A}) : Y \subseteq X\}$.

Lemma 6.4. *Let $X \in \mathcal{S}(\mathcal{A})_{D \setminus V}$. Then*

- (1) $\text{rad}(X) = \{e, q, q^2\}$, $q \in E \setminus C_1$;
- (2) X regular and $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$.

Proof. Since $X \not\subseteq U$, we have $\text{rad}(X) > e$. The group $\text{rad}(X)$ does not contain elements of order greater than 3 because by the above assumption $c_1 \notin \text{rad}(X)$. Therefore $\text{rad}(X)$ is a group of order 3 distinct from C_1 and Statement (1) holds. Denote $\text{rad}(X) = \{e, q, q^2\}$ by L . Then $\text{rad}(\pi(X)) = e$, where $\pi : D \rightarrow D/L$ is the quotient epimorphism. The group D/L is cyclic. Thus by Lemma 2.8, the set $\pi(X)$ is regular or $\pi(X) = H \setminus \{e\}$ for some $H \leq D/L$. If $\pi(X) = H \setminus \{e\}$ then X contains all elements of order 3 distinct from q and q^2 . Hence L is a unique minimal \mathcal{A} -subgroup, a contradiction. So $\pi(X)$ and X are regular. Since $LX = X$, we conclude that $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. \square

Statement (2) of Lemma 6.4 yields that

$$\bigcup_{X \in \mathcal{S}(\mathcal{A})_{D \setminus V}} \text{tr}(X) = D \setminus D_k$$

for some $k \geq 1$. So $V = D_k$ is an \mathcal{A} -subgroup.

Lemma 6.5. *Let $X, Y \in \mathcal{S}(\mathcal{A})_{D \setminus V}$. Then $\text{rad}(X) = \text{rad}(Y)$.*

Proof. Assume the contrary. Then by Lemma 6.4, without loss of generality we may assume that $\text{rad}(X) = \{e, s, s^2\} = S$ and $\text{rad}(Y) = \{e, sc_1, sc_1^2\}$. So

$$X = X_{0e} \cup sX_{0e} \cup s^2X_{0e}, \quad Y = Y_{0e} \cup sc_1Y_{0e} \cup s^2c_1^2Y_{0e}.$$

The sets X_{0e} and Y_{0e} are regular and nonempty by Statement (2) of Lemma 6.4. Thus X_{0e} and Y_{0e} are orbits of a group $K \leq \text{Aut}(C)$. Furthermore, $\text{rad}(X_{0e}) = \text{rad}(Y_{0e}) = e$ since otherwise $\text{rad}(X)$ or $\text{rad}(Y)$ contains at least nine elements that contradicts Statement (1) of Lemma 6.4. Consider the basic sets $\pi(X)$ and $\pi(Y)$ of the circulant S -ring $\mathcal{A}_{D/S}$, where $\pi : D \rightarrow D/S$ is the quotient epimorphism. The radical of $\pi(X) = X_{0e}$ is trivial, the radical $\pi(Y)$ contains $\pi(sc_1)$. This contradicts Lemma 2.9 applied for an S -ring $\pi(\mathcal{A}_{(X)})$ and \mathcal{A} -section $\pi(\langle Y \rangle)$. \square

Lemma 6.6. $\text{rad}(\mathcal{A}_V) = e$. In particular, $U = V$.

Proof. Let X be the highest basic set of \mathcal{A}_V . Since $V \neq D$, there exists $Y \in \mathcal{S}(\mathcal{A})_{D \setminus V}$ such that $X \cap Y^3 \neq \emptyset$. Moreover, $X \subset Y^3$ because Y^3 is an \mathcal{A} -set. By Lemma 6.4, without loss of generality we may assume that $Y = Y_{0e} \cup sY_{0e} \cup s^2Y_{0e}$, $\text{rad}(Y) = S$, $\text{rad}(Y_{0e}) = e$. Then $Y^3 = SY_{0e}^3$. Since Y_{0e} is nonempty and regular by Lemma 6.4, the set Y_{0e} is an orbit of $K \leq \text{Aut}(C)$. Therefore Lemma 2.11 implies that either $Y_{0e} = \{y\}$ or $Y_{0e} = \{y, y^{-1}\}$. So Y^3 is one of two types:

$$\begin{aligned} & \{y^3, sy^3, s^2y^3\}, \\ & \{y, sy, s^2y, y^{-1}, sy^{-1}, s^2y^{-1}, y^3, sy^3, s^2y^3, y^{-3}, sy^{-3}, s^2y^{-3}\}. \end{aligned}$$

Since $X \subset V$, we have $\text{rad}(X) = e$ or $c_1 \in \text{rad}(X)$. Let us show that the latter is impossible. This is obvious if Y^3 is of the first type. If Y^3 is of the second type and $c_1 \in \text{rad}(X)$ then it is easy to check that $y = c_1$ or $y = c_1^2$. However, $y \notin V$. This contradicts the fact that $E \subseteq V$. Thus $\text{rad}(X) = e$ and $\text{rad}(\mathcal{A}_V) = e$. \square

To complete the proof of Theorem 6.1, let $L = \text{rad}(X)$, $X \in \mathcal{S}(\mathcal{A})_{D \setminus V}$. By Lemma 6.4, the group L does not depend on the choice of X . Then \mathcal{A} is the S -wreath product where $S = V/L$. Since $\text{rad}(\mathcal{A}_V) = e$ and $|L| = 3$ the statement of Theorem 6.1 holds. \square

7. S -RINGS OVER $D = \mathbb{Z}_3 \times \mathbb{Z}_{3^n}$: SCHURITY

In this section we prove Theorem 1.1.

Proof of the Theorem 1.1. We proceed by induction on n . The statement of the theorem for $n \leq 3$ follows from calculations in the group ring of D that made by [5]. Let $n \geq 4$ and \mathcal{A} be an S -ring over D . Let us show that \mathcal{A} is schurian. If $\text{rad}(\mathcal{A}) = e$ then \mathcal{A} is schurian by Theorem 4.1 and Theorem 5.1. Suppose that $\text{rad}(\mathcal{A}) > e$. Then Theorem 6.1 implies that \mathcal{A} is the S -wreath product, where $S = U/L$, and one of the following statements holds:

- (1) $|S| = 1$,
- (2) $|S| = 3$,
- (3) $\text{rad}(\mathcal{A}_U) = e$ and $|L| = 3$.

Let us show that in these cases \mathcal{A} is schurian. By the induction hypothesis, the S -rings \mathcal{A}_U and $\mathcal{A}_{D/L}$ are schurian. So by Corollary 2.1, it is sufficient to prove that $\text{Aut}(\mathcal{A}_S)$ is 2-isolated. Obviously, the group $\text{Aut}(\mathcal{A}_S)$ is 2-isolated whenever $|S| = 1$ or $|S| = 3$. Hence we may assume that $|S| \geq 9$. Let $\text{rad}(\mathcal{A}_U) = e$ and $|L| = 3$. If U is cyclic then Lemma 2.8 and Lemma 2.11 imply that every basic set of \mathcal{A}_U is of the form $\{x\}$ or every basic set of \mathcal{A}_U is of the form $\{x, x^{-1}\}$. In these cases, obviously, $\text{Aut}(\mathcal{A}_{U/L})_e$ has a faithful regular orbit. So $\text{Aut}(\mathcal{A}_{U/L})$ is 2-isolated by Lemma 2.15. Thus we may assume that $U = D_k$, where $k < n$. Since L is an \mathcal{A} -subgroup of order 3, we conclude that $\text{rk}(\mathcal{A}_U) > 2$.

Suppose that \mathcal{A}_U is not regular. Then Theorem 4.1 implies that $\mathcal{A}_U = \mathcal{A}_H \otimes \mathcal{A}_L$, where $\text{rk}(\mathcal{A}_H) = 2$.

If $\text{rk}(\mathcal{A}_L) = 2$, we have

$$\text{Aut}(\mathcal{A}_U)^S = (\text{Sym}(H) \times \text{Sym}(L))^{U/L} = \text{Sym}(U/L).$$

If $\text{rk}(\mathcal{A}_L) = 3$, we have

$$\text{Aut}(\mathcal{A}_U)^S = (\text{Sym}(H) \times L_{\text{right}})^{U/L} = \text{Sym}(U/L).$$

Note that D/L is cyclic and S is an $\mathcal{A}_{D/L}$ -section of composite order. From [1, Theorem 4.6] it follows that $\text{Aut}(\mathcal{A}_{D/L})^S = \text{Sym}(U/L)$. Thus \mathcal{A} is schurian by Theorem 2.4 applied for $\Delta_0 = \text{Aut}(\mathcal{A}_{D/L})$ and

$\Delta_1 = \text{Aut}(\mathcal{A}_U)$. Hence by Theorem 4.1, we may assume that \mathcal{A}_U is regular. From Theorem 5.1 it follows that $\mathcal{A}_U = \text{Cyc}(K, U)$, where K listed in Table 1. If $\mathcal{A}_U = \text{Cyc}(K_i, U)$, $i \in \{0, 1, 2, 3, 4, 5\}$, then the group $\mathcal{A}_{U/L}$ is 2-isolated by Corollary 5.2.

Let $\mathcal{A}_U = \text{Cyc}(K_i, U)$, $i \in \{6, 7, 8, 9\}$. Then C_1 is the unique \mathcal{A} -subgroup of order 3 and, hence, $L = C_1$. Besides, for every highest basic set X of \mathcal{A}_U we have $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. Let us show that the group $H = \text{rad}(\mathcal{A})$ is not cyclic. Assume the contrary. Note that $C_1 \leq H < U$. The first inequality holds because otherwise H is an \mathcal{A} -subgroup of order 3 distinct from C_1 . The latter inequality holds because otherwise $X \subseteq H$ for some highest basic set X of \mathcal{A}_U and H can not be cyclic since $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$. Next, the group D/H is isomorphic to D_l , where $l < n$. The S -ring $\mathcal{A}_{D/H}$ has trivial radical because otherwise H is not the radical of \mathcal{A} . The group $\tilde{E} = \{x \in D/H : |x| = 3\}$ is an $\mathcal{A}_{D/H}$ -subgroup because every basic set of $\mathcal{A}_{U/H}$ is regular and U/H contains all elements of order 3. Theorem 4.1 applied to $D/H \cong D_l$ yields that $\mathcal{A}_{D/H}$ is regular. Then by Corollary 5.1, there exists a cyclic $\mathcal{A}_{D/H}$ -subgroup of order 3^{l-1} . This contradicts the fact that $X_{0e} \neq \emptyset$, $X_{1s} \neq \emptyset$, $X_{2s^2} \neq \emptyset$ for all highest basic sets of $\mathcal{A}_{U/H}$. Therefore H is not cyclic and, hence, D/H is cyclic.

Since H is not cyclic, there exists a highest basic set X of \mathcal{A} such that $s \in \text{rad}(X)$. By Theorem 2.2, every highest basic set of \mathcal{A} is rationally conjugate to X and $\text{rad}(Y) = H$ for every highest basic set Y of \mathcal{A} . Therefore $\mathcal{A}_{D/H}$ is circulant and $\text{rad}(\mathcal{A}_{D/H}) = e$. Lemma 2.8 and Lemma 2.11 imply that every basic set of $\mathcal{A}_{D/H}$ is of the form $\{x\}$ or every basic set of $\mathcal{A}_{D/H}$ is of the form $\{x, x^{-1}\}$. So \mathcal{A} is regular and D_{n-1} is an \mathcal{A} -subgroup. Since $H \leq D_{n-1}$ and $\text{rad}(Y) = H$ for every highest basic set Y of \mathcal{A} , we conclude that \mathcal{A} is an D_{n-1}/H -wreath product. By the induction hypothesis, S -rings $\mathcal{A}_{D_{n-1}}$ and $\mathcal{A}_{D/H}$ are schurian. The basic sets of circulant S -ring $\mathcal{A}_{D_{n-1}/H}$ are of the form $\{x\}$ or of the form $\{x, x^{-1}\}$. Thus $\text{Aut}(\mathcal{A}_{D_{n-1}/H})_e$ has a faithful regular orbit. So $\text{Aut}(\mathcal{A}_{D_{n-1}/H})$ is 2-isolated by Lemma 2.15. Therefore Corollary 2.1 applied to $\tilde{S} = D_{n-1}/H$ yields that \mathcal{A} is schurian which completes the proof of the theorem. \square

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REFERENCES

- [1] *S. Evdokimov, I. Ponomarenko*, Schurity of S -rings over a cyclic group and generalized wreath product of permutation groups, *Algebra and Analysis*, 24 (2012), 3, 84-127.
- [2] *S. Evdokimov, I. Ponomarenko*, Characterization of cyclotomic schemes and normal Schur rings over a cyclic group, *Algebra and Analysis*, 14 (2002), 2, 11-55.
- [3] *S. Evdokimov, I. Kovács, I. Ponomarenko*, Characterization of cyclic Schur groups, *Algebra and Analysis*, 25 (2013), 61-85.
- [4] *S. Evdokimov, I. Kovács, I. Ponomarenko*, On schurity of finite abelian groups, *Communications Algebra*, 44 (2016), 1, 101–117.
- [5] *The GAP Group*, *GAP Groups, Algorithms, and Programming*, Version 4.7.6; 2014, <http://www.gap-system.org>.
- [6] *M. Klin, R. Pöschel*, The isomorphism problem for circulant digraphs with p^n vertices, Preprint P-34/80 Akad. der Wiss. der DDR, ZIMM, Berlin, 1980.
- [7] *M. Muzychuk, I. Ponomarenko*, On Schur 2-groups, *Zapiski Nauchnykh Seminarov POMI*, 435 (2015), 113-162.
- [8] *I. Ponomarenko, A. Vasil'ev*, On non-abelian Schur groups, *Journal of Algebra and Its Applications*, 13 (2014), 8, 1450055-1-1450055-22.
- [9] *R. Pöschel*, Untersuchungen von S -Ringen insbesondere im Gruppenring von p -Gruppen, *Math. Nachr.*, 60 (1974), 1-27.
- [10] *G. Ryabov*, On Schur 3-groups, *Siberian Electronic Mathematical Reports*, 12 (2015) 223-331.
- [11] *I. Schur*, Zur theorie der einfach transitiven Permutationengruppen, *S.-B. Preus Akad. Wiss. Phys.-Math. Kl.* (1933), 598-623.
- [12] *H. Wielandt*, *Finite permutation groups*, Academic Press, New York - London, 1964.
- [13] *H. Wielandt*, *Permutation groups through invariant relations and invariant functions*, *Lecture Notes Dept. Math.*, Ohio State Univ., Columbus, Ohio, 1969.

NOVOSIBIRSK STATE UNIVERSITY, 2 PIROGOVA ST., 630090 NOVOSIBIRSK, RUSSIA
E-mail address: `gric2ryabov@gmail.com`